# MODULE-V

## CHAPTER-5 (5.1, 5.2, 5.3, 5.4, 5.5, 5.6, 5.7)

# LAPLACE TRANSFORM

## BASIC FORMULAS

1. $\sin(A+B) = \sin A \cos B + \cos A \sin B$
2. $\sin(A-B) = \sin A \cos B - \cos A \sin B$
3. $\cos(A+B) = \cos A \cos B - \sin A \sin B$
4. $\cos(A-B) = \cos A \cos B + \sin A \sin B$
5. $\cos 2t = 2\cos^2 t - 1 = 1 - 2\sin^2 t$
6. $\sin 2t = 2\sin t \cos t$
7. $\cosh 2t = 2\cosh^2 t - 1 = 1 + 2\sinh^2 t$
8. $\sin 3t = 3\sin t - 4\sin^3 t$
9. $\cos 3t = 4\cos^3 t - 3\cos t$
10. $\cosh at = \dfrac{e^{at} + e^{-at}}{2}$
11. $\sinh at = \dfrac{e^{at} - e^{-at}}{2}$
12. $e^{i\omega t} = \cos \omega t + i \sin \omega t$

    $e^{-i\omega t} = \cos \omega t - i \sin \omega t$
13. $\cos \omega t = \dfrac{e^{i\omega t} + e^{-i\omega t}}{2}$
14. $\sin \omega t = \dfrac{e^{i\omega t} - e^{-i\omega t}}{2i}$

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# 5.1 Laplace Transform, Inverse Transform Linearity, Shifting.

## Definition

Let $f(t)$ be a function which is defined for all $t \geq 0$. Multiply $f(t)$ by $e^{-st}$ integrate w.r.to t from $t=0$ to $t=\infty$. If the resulting integral exists then it will be a function of s say $F(s)$. This $F(s)$ is called the Laplace transform of $f(t)$ written as $L\{f(t)\}$ i.e

$$L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

Since, $L\{f(t)\} = F(s)$

$\Rightarrow f(t) = L^{-1}\{F(s)\}$

$\Rightarrow f(t)$ is the inverse Laplace transform of $F(s)$ where as $F(s)$ is the Laplace transform of $f(t)$

NOTE:-

(i) t - variable — Laplace (L) — S-variable

(ii) S - variable — Inverse Laplace ($L^{-1}$) — t-variable.

## Laplace transform of some standard functions

(i) Let $f(t)=1$

$$L\{f(t)\} = L\{1\} = \int_0^\infty e^{-st}\,1\,dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty = -\frac{1}{s}[0-1] = \frac{1}{s}$$

So $L\{1\} = \frac{1}{s}$.

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Let $f(t) = t$

$$L\{f(t)\} = \int_0^\infty e^{-st}\, t\, dt = \left(t\,\frac{e^{-st}}{-s}\right)_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}\cdot 1\, dt$$

$$= 0 + \frac{1}{s}\int_0^\infty e^{-st}\, 1\, dt = \frac{1}{s}\cdot\frac{1}{s} = \frac{1}{s^2}$$

So, $L\{t\} = \frac{1}{s^2}$

Let $f(t) = t^2$

$$L\{f(t)\} = L\{t^2\} = \int_0^\infty e^{-st}\, t^2\, dt = \left(t^2\,\frac{e^{-st}}{-s}\right)_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}\, 2t\, dt$$

$$= 0 + \frac{2}{s}\int_0^\infty e^{-st}\, t\, dt$$

$$= \frac{2}{s}\cdot\frac{1}{s^2} = \frac{2}{s^3}$$

So, $L\{t^2\} = \frac{2}{s^3}$

Let $f(t) = t^3$

$$L\{f(t)\} = L\{t^3\} = \int_0^\infty e^{-st}\, t^3\, dt = \left(t^3\,\frac{e^{-st}}{-s}\right)_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}\, 3t^2\, dt$$

$$= 0 + \frac{3}{s}\int_0^\infty e^{-st}\, t^2\, dt$$

$$= \frac{3}{s}\cdot\frac{2}{s^3} = \frac{6}{s^4}$$

So, $L\{t^3\} = \frac{6}{s^4}$

Thus we find that

$$L\{1\} = \frac{1}{s} = \frac{0!}{s^{0+1}}$$

$$L\{t\} = \frac{1}{s^2} = \frac{1!}{s^{1+1}}$$

$$L\{t^2\} = \frac{2}{s^3} = \frac{2!}{s^{2+1}}$$

$$L\{t^3\} = \frac{6}{s^4} = \frac{3!}{s^{3+1}}$$

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In general,

$$L\{t^n\} = \frac{n!}{s^{n+1}}, \quad n = 0, 1, 2, 3, 4 \cdots$$

Example-

$$L\{t^4\} = \frac{4!}{s^{4+1}} = \frac{24}{s^5}$$

$$L\{t^5\} = \frac{5!}{s^{5+1}} = \frac{120}{s^6}$$

$$L\{t^6\} = \frac{6!}{s^{6+1}} = \frac{720}{s^7}.$$

(ii) Let $f(t) = e^{at}$

$$L\{f(t)\} = L\{e^{at}\} = \int_0^\infty e^{-st} e^{at} dt$$

$$= \int_0^\infty e^{-(s-a)t} dt = \left[ \frac{e^{-(s-a)t}}{-(s-a)} \right]_0^\infty$$

$$= -\frac{1}{(s-a)} [0 - 1] = \frac{1}{s-a}$$

Thus $\boxed{L\{e^{at}\} = \frac{1}{s-a}}$

Ex:- $L\{e^t\} = \frac{1}{s-1}$

$$L\{e^{2t}\} = \frac{1}{s-2}$$

$$L\{e^{3t}\} = \frac{1}{s-3}.$$

(iii) Let $f(t) = e^{-at}$

$$L\{f(t)\} = L\{e^{-at}\} = \int_0^\infty e^{-st} e^{-at} dt = \int_0^\infty e^{-(s+a)t} dt$$

$$= \left[ \frac{e^{-(s+a)t}}{-(s+a)} \right]_0^\infty$$

$$= -\frac{1}{(s+a)} [0 - 1] = \frac{1}{s+a}$$

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Thus, $\boxed{L\{e^{-at}\} = \frac{1}{s+a}}$

Ex:- $L\{e^{-t}\} = \frac{1}{s+1}$

$L\{e^{-2t}\} = \frac{1}{s+2}$

$L\{e^{-3t}\} = \frac{1}{s+3}$.

(IV) Let $f(t) = \sin \omega t$

$$L\{f(t)\} = \int_0^\infty e^{-st} \sin \omega t \, dt$$

$$= \int_0^\infty e^{-st} \left( \frac{e^{i\omega t} - e^{-i\omega t}}{2i} \right) dt$$

$$= \frac{1}{2i} \int_0^\infty \left[ e^{-(s-i\omega)t} - e^{-(s+i\omega)t} \right] dt$$

$$= \frac{1}{2i} \left[ \frac{e^{-(s-i\omega)t}}{-(s-i\omega)} - \frac{e^{-(s+i\omega)t}}{-(s+i\omega)} \right]_0^\infty$$

$$= \frac{1}{2i} \left[ \frac{1}{-(s-i\omega)} (0-1) - \frac{1}{-(s+i\omega)} (0-1) \right]$$

$$= \frac{1}{2i} \left[ \frac{1}{s-i\omega} - \frac{1}{s+i\omega} \right]$$

$$= \frac{1}{2i} \left[ \frac{s+i\omega - s + i\omega}{(s-i\omega)(s+i\omega)} \right] = \frac{1}{2i} \left[ \frac{2i\omega}{s^2 - i^2\omega^2} \right]$$

$$= \frac{\omega}{s^2+\omega^2}$$

Thus, $\boxed{L\{\sin \omega t\} = \frac{\omega}{s^2+\omega^2}}$

Ex:- $L\{\sin t\} = \frac{1}{s^2+1}$

$L\{\sin 2t\} = \frac{2}{s^2+4}$

$L\{\sin 3t\} = \frac{3}{s^2+9}$

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(V) Let $f(t) = \cos\omega t$

$$L\{f(t)\} = L\{\cos\omega t\} = \int_0^\infty e^{-st}\cos\omega t\,dt$$

$$= \int_0^\infty e^{-st}\left(\frac{e^{i\omega t}+e^{-i\omega t}}{2}\right)dt$$

$$= \frac{1}{2}\int_0^\infty \left[e^{-(s-i\omega)t} + e^{-(s+i\omega)t}\right]dt$$

$$= \frac{1}{2}\left[\frac{e^{-(s-i\omega)t}}{-(s-i\omega)} + \frac{e^{-(s+i\omega)t}}{-(s+i\omega)}\right]_0^\infty$$

$$= \frac{1}{2}\left[\frac{1}{-(s-i\omega)}(0-1) + \frac{1}{-(s+i\omega)}(0-1)\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-i\omega} + \frac{1}{s+i\omega}\right]$$

$$= \frac{1}{2}\left[\frac{s+i\omega+s-i\omega}{(s-i\omega)(s+i\omega)}\right] = \frac{1}{2}\left[\frac{2s}{s^2-i^2\omega^2}\right]$$

$$= \frac{s}{s^2+\omega^2}$$

Thus, $\boxed{L\{\cos\omega t\} = \frac{s}{s^2+\omega^2}}$

Ex:- $L\{\cos t\} = \frac{s}{s^2+1}$

$L\{\cos 2t\} = \frac{s}{s^2+4}$

$L\{\cos 3t\} = \frac{s}{s^2+9}$.

(VI) Let $f(t) = \sinh at$

$$L\{f(t)\} = L\{\sinh at\} = \int_0^\infty e^{-st}\sinh at\,dt$$

$$= \int_0^\infty e^{-st}\left(\frac{e^{at}-e^{-at}}{2}\right)dt$$

$$= \frac{1}{2}\int_0^\infty \left[e^{-(s-a)t} - e^{-(s+a)t}\right]dt$$

$$= \frac{1}{2}\left[\frac{e^{-(s-a)t}}{-(s-a)} - \frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty$$

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$$= \frac{1}{2}\left[\frac{1}{-(s-a)}(0-1) - \frac{1}{-(s+a)}(0-1)\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] = \frac{1}{2}\left[\frac{s+a-s+a}{(s-a)(s+a)}\right]$$

$$= \frac{1}{2}\left[\frac{2a}{s^2-a^2}\right] = \frac{a}{s^2-a^2}$$

Thus, $\boxed{L\{\sinh at\} = \frac{a}{s^2-a^2}}$

Example:-

$$L\{\sinh t\} = \frac{1}{s^2-1}$$

$$L\{\sinh 2t\} = \frac{2}{s^2-4}$$

$$L\{\sinh 3t\} = \frac{3}{s^2-9}.$$

(VII) Let $f(t) = \cosh at$

$$L\{f(t)\} = L\{\cosh at\} = \int_0^\infty e^{-st}\cosh at\, dt$$

$$= \int_0^\infty e^{-st}\left(\frac{e^{at}+e^{-at}}{2}\right)dt$$

$$= \frac{1}{2}\int_0^\infty \left[e^{-(s-a)t} + e^{-(s+a)t}\right]dt$$

$$= \frac{1}{2}\left[\frac{e^{-(s-a)t}}{-(s-a)} + \frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty$$

$$= \frac{1}{2}\left[\frac{1}{-(s-a)}(0-1) + \frac{1}{-(s+a)}(0-1)\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right] = \frac{1}{2}\left[\frac{s+a+s-a}{(s-a)(s+a)}\right]$$

$$= \frac{1}{2}\left[\frac{2s}{s^2-a^2}\right] = \frac{s}{s^2-a^2}$$

$\boxed{L\{\cosh at\} = \frac{s}{s^2-a^2}}$

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Ex:- $L\{\cosh t\} = \frac{s}{s^2-1}$

$L\{\cosh 2t\} = \frac{s}{s^2-4}$

$L\{\cosh 3t\} = \frac{s}{s^2-9}$.

Thus, in general we find that

(i) $L\{t^n\} = \frac{n!}{s^{n+1}}$, $n = 0, 1, 2 \ldots\ldots$

(II) $L\{e^{at}\} = \frac{1}{s-a}$

(III) $L\{e^{-at}\} = \frac{1}{s+a}$

(IV) $L\{\sin \omega t\} = \frac{\omega}{s^2+\omega^2}$

(V) $L\{\cos \omega t\} = \frac{s}{s^2+\omega^2}$

(VI) $L\{\sinh at\} = \frac{a}{s^2-a^2}$

(VII) $L\{\cosh at\} = \frac{s}{s^2-a^2}$.

NOTE:- we have Laplace transform (L) is a linear operator.

(i) $L\{f(t)+g(t)\} = L\{f(t)\} + L\{g(t)\}$

(II) $L\{f(t)-g(t)\} = L\{f(t)\} - L\{g(t)\}$

(III) $L\{Kf(t)\} = K\,L\{f(t)\}$

NOTE:- These two properties does not hold in Laplace transform

(I) $L\{f(t)\,g(t)\} \neq L\{f(t)\}\,L\{g(t)\}$

(II) $L\left\{\frac{f(t)}{g(t)}\right\} \neq \frac{L\{f(t)\}}{L\{g(t)\}}$

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## Example

Find the Laplace transform of

$$f(t) = t^2 + 4e^{2t} - 3\sinh t + 8\cos 2t$$

Ans:-

$$L\{f(t)\} = L\{t^2 + 4e^{2t} - 3\sinh t + 8\cos 2t\}$$

$$= L\{t^2\} + L\{4e^{2t}\} - L\{3\sinh t\} + L\{8\cos 2t\}$$

$$= L\{t^2\} + 4L\{e^{2t}\} - 3L\{\sinh t\} + 8L\{\cos 2t\}$$

$$= \frac{2}{s^3} + 4\frac{1}{s-2} - 3\frac{1}{s^2-1} + 8\frac{s}{s^2+4}$$

$$= \frac{2}{s^3} + \frac{4}{s-2} - \frac{3}{s^2-1} + \frac{8s}{s^2+4}$$

## Properties of gama ($\Gamma$) function

(i) $\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx$

(ii) $\Gamma 1 = 1$

(iii) $\Gamma\frac{1}{2} = \sqrt{\pi}$

(iv) $\Gamma$(-ve number, zero) $\rightarrow$ does not exist

(v) $\Gamma n = (n-1)\Gamma(n-1)$

$= (n-1)(n-2)\Gamma(n-2)$

$= (n-1)(n-2)\cdots 3\cdot 2\cdot 1\,\Gamma 1$

$= (n-1)(n-2)\cdots 3\cdot 2\cdot 1\cdot 1$

$= (n-1)!$

So, $\boxed{\Gamma n = (n-1)!}$

## Result-1

If $n > -1$, may be a fraction

$$L\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}$$

## Example:-

Find the Laplace transform of

(i) $t^{-\frac{1}{2}}$ (ii) $t^{3/2}$

## Ans to (i)

$$L\{t^{-\frac{1}{2}}\} = \frac{\Gamma(-\frac{1}{2}+1)}{s^{-\frac{1}{2}+1}} = \frac{\Gamma\frac{1}{2}}{s^{\frac{1}{2}}} = \frac{\sqrt{\pi}}{\sqrt{s}} = \sqrt{\frac{\pi}{s}}$$

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Ans to (ii)

$$L\{t^{3/2}\} = \frac{\Gamma(\frac{3}{2}+1)}{s^{3/2+1}} = \frac{\Gamma\frac{5}{2}}{s^{5/2}} = \frac{\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}}{s^{5/2}} = \frac{3\sqrt{\pi}}{4s^{5/2}}.$$

Example

Find the Laplace transform of

(i) $\pi^t$ (ii) $3^{-t}$

Ans to (i)

$$L\{\pi^t\} = L\{e^{\ln \pi^t}\} \qquad (\because e^{\ln x} = x)$$

$$= L\{e^{t\ln\pi}\} \qquad (\because \ln a^b = b\ln a)$$

$$= \frac{1}{s-\ln\pi}$$

Ans to (ii)

$$L\{3^{-t}\} = L\{e^{\ln 3^{-t}}\} = L\{e^{-t\ln 3}\} = \frac{1}{s+\ln 3}.$$

Inverse Laplace Transform

(i) $L\{t^n\} = \frac{n!}{s^{n+1}} \Rightarrow \boxed{L^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n, \quad n=0,1,2\cdots}$

Ex: $L^{-1}\left\{\frac{24}{s^5}\right\} = L^{-1}\left\{\frac{4!}{s^{4+1}}\right\} = t^4$

$$L^{-1}\left\{\frac{8}{s^5}\right\} = \frac{1}{3}L^{-1}\left\{\frac{4!}{s^{4+1}}\right\} = \frac{1}{3}t^4$$

$$L^{-1}\left\{\frac{48}{s^5}\right\} = 2L^{-1}\left\{\frac{4!}{s^{4+1}}\right\} = 2t^4$$

$$L^{-1}\left\{\frac{7}{s^3}\right\} = \frac{7}{2}L^{-1}\left\{\frac{2!}{s^{2+1}}\right\} = \frac{7}{2}t^2$$

$$L^{-1}\left\{\frac{2}{s^4}\right\} = \frac{1}{3}L^{-1}\left\{\frac{3!}{s^{3+1}}\right\} = \frac{1}{3}t^3.$$

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(ii) $L\{e^{at}\} = \frac{1}{s-a} \Rightarrow \boxed{L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}}$

Ex: $L^{-1}\left\{\frac{1}{s-4}\right\} = e^{4t}$

$L^{-1}\left\{\frac{3}{s-4}\right\} = 3e^{4t}$

(III) $L\{e^{-at}\} = \frac{1}{s+a} \Rightarrow \boxed{L^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}}$

Ex: $L^{-1}\left\{\frac{1}{s+4}\right\} = e^{-4t}$

$L^{-1}\left\{\frac{2}{s+4}\right\} = 2e^{-4t}$

(IV) $L\{\sin\omega t\} = \frac{\omega}{s^2+\omega^2} \Rightarrow \boxed{L^{-1}\left\{\frac{\omega}{s^2+\omega^2}\right\} = \sin\omega t}$

Ex: $L^{-1}\left\{\frac{4}{s^2+16}\right\} = \sin 4t$

$L^{-1}\left\{\frac{8}{s^2+16}\right\} = 2\sin 4t$

$L^{-1}\left\{\frac{2}{s^2+16}\right\} = \frac{1}{2}\sin 4t$

$L^{-1}\left\{\frac{3}{s^2+16}\right\} = \frac{3}{4}\sin 4t$

(V) $L\{\cos\omega t\} = \frac{s}{s^2+\omega^2} \Rightarrow \boxed{L^{-1}\left\{\frac{s}{s^2+\omega^2}\right\} = \cos\omega t}$

Ex: $L^{-1}\left\{\frac{s}{s^2+9}\right\} = \cos 3t$

$L^{-1}\left\{\frac{3s}{s^2+9}\right\} = 3\cos 3t$

(VI) $L\{\sinh at\} = \frac{a}{s^2-a^2} \Rightarrow \boxed{L^{-1}\left\{\frac{a}{s^2-a^2}\right\} = \sinh at}$

Ex: $L^{-1}\left\{\frac{4}{s^2-16}\right\} = \sinh 4t$

$L^{-1}\left\{\frac{8}{s^2-16}\right\} = 2\sinh 4t$

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$$L^{-1}\left\{\frac{2}{s^2-16}\right\} = \frac{1}{2}\sinh 4t$$

$$L^{-1}\left\{\frac{3}{s^2-16}\right\} = \frac{3}{4}\sinh 4t$$

(VII) $L\{\cosh at\} = \frac{s}{s^2-a^2} \Rightarrow \boxed{L^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at}$

Ex:

$$L^{-1}\left\{\frac{s}{s^2-9}\right\} = \cosh 3t$$

$$L^{-1}\left\{\frac{2s}{s^2-9}\right\} = 2\cosh 3t.$$

Thus in general we find that-

(i) $L^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n, \quad n = 0, 1, 2 \ldots$

(II) $L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$

(III) $L^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}$

(IV) $L^{-1}\left\{\frac{w}{s^2+w^2}\right\} = \sin wt$

(V) $L^{-1}\left\{\frac{s}{s^2+w^2}\right\} = \cos wt$

(VI) $L^{-1}\left\{\frac{a}{s^2-a^2}\right\} = \sinh at$

(VII) $L^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at$

Ex:-

(i) $L^{-1}\left\{\frac{2s}{s^2+9}\right\} = 2\cos 3t$

(II) $L^{-1}\left\{\frac{4}{s^2-16}\right\} = \sinh 4t$

(III) $L^{-1}\left\{\frac{2}{s-1}\right\} = 2e^{t}$

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**NOTE:-** Like Laplace transform ($L$) Inverse Laplace transform ($L^{-1}$) is also a linear operator.

(i) $L^{-1}\{F(s)+G(s)\} = L^{-1}\{F(s)\} + L^{-1}\{G(s)\}$

(ii) $L^{-1}\{F(s)-G(s)\} = L^{-1}\{F(s)\} - L^{-1}\{G(s)\}$

(iii) $L^{-1}\{KF(s)\} = K\,L^{-1}\{F(s)\}$

**Example:-**

Find the inverse Laplace transform of

$$F(s) = \frac{2}{s^4} - \frac{4s}{s^2+9} + \frac{3}{s^2-25}$$

**Ans:**

~~$L^{-1}\{F(s)\} \equiv L^{-1}\left\{\frac{2}{s^4}\right\}$~~

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{2}{s^4} - \frac{4s}{s^2+9} + \frac{3}{s^2-25}\right\}$$

$$= L^{-1}\left\{\frac{2}{s^4}\right\} - L^{-1}\left\{\frac{4s}{s^2+9}\right\} + L^{-1}\left\{\frac{3}{s^2-25}\right\}$$

$$= \frac{1}{3}L^{-1}\left\{\frac{3!}{s^{3+1}}\right\} - 4\,L^{-1}\left\{\frac{s}{s^2+9}\right\} + \frac{3}{5}L^{-1}\left\{\frac{5}{s^2-25}\right\}$$

$$= \frac{1}{3}t^3 - 4\cos 3t + \frac{3}{5}\sinh 5t.$$

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## Example

Find the Laplace Transform of

(i) $\sin t \cos t$ (ii) $\cos^2 \frac{1}{2}t$ (iii) $\sinh^2 2t$ (iv) $\sin^3 t$

### Ans to (i)

$$L\{\sin t \cos t\} = \frac{1}{2} L\{2 \sin t \cos t\} = \frac{1}{2} L\{\sin 2t\}$$

$$= \frac{1}{2} \cdot \frac{2}{s^2+4} = \frac{1}{s^2+4}$$

### Ans to (ii)

$$L\{\cos^2 \tfrac{1}{2}t\} = L\left\{\frac{1+\cos t}{2}\right\} = \frac{1}{2} L\{1+\cos t\}$$

$$= \frac{1}{2}\left[L\{1\} + L\{\cos t\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+1}\right]$$

### Ans to (iii)

$$L\{\sinh^2 2t\} = L\left\{\frac{\cosh 4t - 1}{2}\right\} = \frac{1}{2} L\{\cosh 4t - 1\}$$

$$= \frac{1}{2}\left[L\{\cosh 4t\} - L\{1\}\right]$$

$$= \frac{1}{2}\left[\frac{s}{s^2+16} - \frac{1}{s}\right]$$

### Ans to (iv)

$$L\{\sin^3 t\} = L\left\{\frac{3\sin t - \sin 3t}{4}\right\}$$

$$= L\left\{\frac{3}{4}\sin t - \frac{1}{4}\sin 3t\right\}$$

$$= \frac{3}{4} L\{\sin t\} - \frac{1}{4} L\{\sin 3t\}$$

$$= \frac{3}{4} \cdot \frac{1}{s^2+4} - \frac{1}{4} \frac{3}{s^2+9}.$$

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# FIRST SHIFTING THEOREM

Let $L\{f(t)\} = F(s)$

Then $L\{e^{at} f(t)\} = F(s-a)$

$L\{e^{-at} f(t)\} = F(s+a)$

Proof:-

It is given that

$$F(s) = L\{f(t)\}$$

$$= \int_0^\infty e^{-st} f(t)\, dt$$

Then $$F(s-a) = \int_0^\infty e^{-(s-a)t} f(t)\, dt$$

$$= \int_0^\infty e^{-st+at} f(t)\, dt$$

$$= \int_0^\infty e^{-st} e^{at} f(t)\, dt$$

$$= L\{e^{at} f(t)\}$$

$$\Rightarrow L\{e^{at} f(t)\} = F(s-a)$$

Similarly,

$$F(s+a) = \int_0^\infty e^{-(s+a)t} f(t)\, dt$$

$$= \int_0^\infty e^{-st-at} f(t)\, dt$$

$$= \int_0^\infty e^{-st} e^{-at} f(t)\, dt$$

$$= L\{e^{-at} f(t)\}$$

$$\Rightarrow L\{e^{-at} f(t)\} = F(s+a)$$

(Proved)

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NOTE:-

If any function $f(t)$ is multiplied by Possitive or negative power of exponential function then First shifting theorem is applicable

Example

Find the Laplace transform of

(i) $e^{-\alpha t}\cos\beta t$ (ii) $5e^{2t}\sinh 2t$ (iii) $2^t\cos t$

(iv) $2e^{-t}\cos^2\frac{1}{2}t$

Ans to (i)

$$e^{-\alpha t}\cos\beta t$$

Let $f(t) = \cos\beta t$,

$$L\{f(t)\} = L\{\cos\beta t\} = \frac{s}{s^2+\beta^2} = F(s)$$

Thus by First shifting theorem

$$L\{e^{-\alpha t}\cos\beta t\} = F(s+\alpha) = \frac{s+\alpha}{(s+\alpha)^2+\beta^2}$$

Ans to (ii)

$$5e^{2t}\sinh 2t$$

Let $f(t) = 5\sinh 2t$

$$L\{f(t)\} = L\{5\sinh 2t\} = 5\frac{2}{s^2-4} = \frac{10}{s^2-4} = F(s)$$

Thus by F.S.T

$$L\{5e^{2t}\sinh 2t\} = F(s-2) = \frac{10}{(s-2)^2-4}$$

Ans to (iii)

$$2^t\cos t = e^{\ln 2^t}\cos t = e^{t\ln 2}\cos t$$

Let $f(t) = \cos t$

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$$L\{f(t)\} = L\{\cos t\} = \frac{s}{s^2+1} = F(s)$$

Thus by F.S.T

$$L\{2^t \cos t\} = L\{e^{t \ln 2} \cos t\} = F(s-\ln 2)$$

$$= \frac{s-\ln 2}{(s-\ln 2)^2 + 1}$$

Ans to (iv)

$$2e^{-t} \cos^2 \tfrac{1}{2} t$$

Let $f(t) = 2\cos^2 \frac{1}{2} t$

$$L\{f(t)\} = L\{2\cos^2 \tfrac{1}{2} t\} = L\{1 + \cos t\}$$

$$= L\{1\} + L\{\cos t\}$$

$$= \frac{1}{s} + \frac{s}{s^2+1} = F(s)$$

Thus, by F.S.T

$$L\{2e^{-t} \cos^2 \tfrac{1}{2} t\} = F(s+1) = \frac{1}{s+1} + \frac{s+1}{(s+1)^2+1}$$

## Inverse form of First shifting theorem

Let $L^{-1}\{F(s)\} = f(t)$

Then, $L^{-1}\{F(s-a)\} = e^{at} f(t)$

$$L^{-1}\{F(s+a)\} = e^{-at} f(t).$$

NOTE:- If in $F(s)$ all $s$ is replaced by $s-a$ or $s+a$ then First shifting theorem is applicable for finding out the inverse Laplace transform.

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## Example

Find the inverse Laplace transform of

(i) $\frac{12}{(s-3)^4}$ (ii) $\frac{1}{(s+2)^2-16}$ (iii) $\frac{s-4}{(s-4)^2-36}$

(iv) $\frac{s}{(s+1)^2+4}$

## Ans to (i)

$$\frac{12}{(s-3)^4}$$

Here $a = 3$. Let $F(s) = \frac{12}{s^4}$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{12}{s^4}\right\} = 2L^{-1}\left\{\frac{3!}{s^{3+1}}\right\}$$

$$= 2t^3 = f(t)$$

Thus by F.S.T $e^{3t}.2t^3$

$$L^{-1}\left\{\frac{12}{(s-3)^4}\right\} = e^{3t}\,2t^3 = 2e^{3t}t^3.$$

## Ans to (ii)

$$\frac{1}{(s+2)^2-16}$$

Here $a = 2$, Let $F(s) = \frac{1}{s^2-16}$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s^2-16}\right\} = \frac{1}{4}L^{-1}\left\{\frac{4}{s^2-16}\right\}$$

$$= \frac{1}{4}\sinh 4t = f(t)$$

Thus, by F.S.T

$$L^{-1}\left\{\frac{1}{(s+2)^2-16}\right\} = \frac{1}{4}\cdot e^{-2t}\cdot\frac{1}{4}\sinh 4t$$

$$= \frac{1}{4}e^{-2t}\sinh 4t.$$

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(III) $\dfrac{s-4}{(s-4)^2-16}$

Here, $a=4$, Let $F(s) = \dfrac{s}{s^2-16}$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{s}{s^2-16}\right\} = \cosh 4t = f(t)$$

Thus, by F.S.T

$$L^{-1}\left\{\frac{s-4}{(s-4)^2-16}\right\} = e^{4t}\cosh 4t$$

(IV) $\dfrac{s}{(s+1)^2+4}$

$$L^{-1}\left\{\frac{s}{(s+1)^2+4}\right\} = L^{-1}\left\{\frac{s+1-1}{(s+1)^2+4}\right\}$$

$$= L^{-1}\left\{\frac{s+1}{(s+1)^2+4}\right\} - L^{-1}\left\{\frac{1}{(s+1)^2+4}\right\}$$

$$= e^{-t}\cos t - \frac{1}{2}e^{-t}\sin t$$

Partial fraction (For non repeated & linear factors)

$$\frac{f(s)}{(s-a)(s-b)} = \frac{A}{s-a} + \frac{B}{s-b}.$$

Perfect square

$$s^2+2s+2 = s^2+2\cdot 1\cdot s+1^2-1^2+2 = (s+1)^2+1$$

$$2s^2-3s+1 = 2\left[s^2-\frac{3}{2}s+\frac{1}{2}\right] = 2\left[s^2-2\cdot\frac{3}{4}s+\left(\frac{3}{4}\right)^2 - \left(\frac{3}{4}\right)^2+\frac{1}{2}\right]$$

$$= 2\left[\left(s-\frac{3}{4}\right)^2-\frac{1}{16}\right]$$

$$s^2-5s+5 = s^2-2\cdot\frac{5}{2}\cdot s+\left(\frac{5}{2}\right)^2-\left(\frac{5}{2}\right)^2+5$$

$$= \left(s-\frac{5}{2}\right)^2-\frac{5}{4}.$$

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NOTE:-

Denominator is a second degree Polynomial

- Factorize → APPLY PARTIAL FRACTION
- Perct square → Apply F.S.T

Example

Find the inverse Laplace transform of

(i) $\frac{4}{s^2-2s-3}$ (ii) $\frac{1}{s^2+6s+18}$

Ans to (i)

$$\frac{4}{s^2-2s-3} = \frac{4}{(s-3)(s+1)}$$

By Partial fraction

$$\frac{4}{(s-3)(s+1)} = \frac{A}{s-3} + \frac{B}{s+1}$$

$$\Rightarrow \frac{4}{(s-3)(s+1)} = \frac{A(s+1)+B(s-3)}{(s-3)(s+1)}$$

$$\Rightarrow 4 = A(s+1)+B(s-3)$$

Put $s=3$, $4 = 4A \Rightarrow \boxed{A=1}$

Put $s=-1$, $4 = -4B \Rightarrow \boxed{B=-1}$

Thus, $\frac{4}{s^2-2s-3} = \frac{4}{(s-3)(s+1)} = \frac{1}{s-3} - \frac{1}{s+1}$

$$L^{-1}\left\{\frac{4}{s^2-2s-3}\right\} = L^{-1}\left\{\frac{1}{s-3} - \frac{1}{s+1}\right\} = L^{-1}\left\{\frac{1}{s-3}\right\} - L^{-1}\left\{\frac{1}{s+1}\right\}$$

$$= e^{3t} - e^{-t}.$$

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Ans to (ii)

$$\frac{1}{s^2+6s+18} = \frac{1}{s^2+2\cdot 3\cdot s+3^2-3^2+18} = \frac{1}{(s+3)^2+9}$$

$$L^{-1}\left\{\frac{1}{s^2+6s+18}\right\} = L^{-1}\left\{\frac{1}{(s+3)^2+9}\right\}$$

$$= e^{-3t}\,\frac{1}{3}\sin 3t = \frac{1}{3}e^{-3t}\sin 3t$$

(by F.S-T)

Q:- Stat the existance Theorem of Laplace transform

OR

Write down the existance condition of Laplace transform

Ans:-

Let f(t) be a function i.e piecewise contineous on every finite interval in the range $t \geq 0$ and satisfies

$$|f(t)| \leq Me^{-Kt}$$

for some constants S and K

Then the Laplace transform exists for all $S > K$.

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# Problem Set-5.1, Page-257

E. Kreyszig (8TH EDITION)

NO-1-TO-8

Find the Laplace transform of the following.

2. $a+bt+ct^2$

Ans:-

Let $f(t) = a+bt+ct^2$

$$L\{f(t)\} = L\{a+bt+ct^2\}$$

$$= L\{a\} + L\{bt\} + L\{ct^2\}$$

$$= aL\{1\} + bL\{t\} + cL\{t^2\}$$

$$= a\frac{1}{s} + b\frac{1}{s^2} + c\frac{2}{s^3}$$

$$= \frac{a}{s} + \frac{b}{s^2} + \frac{2c}{s^3}$$

5. $e^{a-bt}$

Ans:- Let $f(t) = e^{a-bt}$

$$L\{f(t)\} = L\{e^{a-bt}\} = L\{e^a e^{-bt}\} = e^a L\{e^{-bt}\}$$

$$= e^a \frac{1}{s+b} = \frac{e^a}{s+b}.$$

7. $\sin(\omega t+\delta)$

Ans:- Let $f(t) = \sin(\omega t+\delta)$

$$L\{f(t)\} = L\{\sin(\omega t+\delta)\}$$

$$= L\{\sin\omega t\cos\delta + \cos\omega t\sin\delta\}$$

$$= L\{\sin\omega t\cos\delta\} + L\{\cos\omega t\sin\delta\}$$

$$= \cos\delta\, L\{\sin\omega t\} + \sin\delta\, L\{\cos\omega t\}$$

$$= \cos\delta\,\frac{\omega}{s^2+\omega^2} + \sin\delta\,\frac{s}{s^2+\omega^2}$$

$$= \frac{\omega\cos\delta + s\sin\delta}{s^2+\omega^2}$$

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17-TO-28

Find the inverse Laplace transform of the following.

18. $\frac{5s}{s^2-25}$

Ans: $L^{-1}\left\{\frac{5s}{s^2-25}\right\} = 5L^{-1}\left\{\frac{s}{s^2-25}\right\} = \cosh 5t$

20. $\frac{s-4}{s^2-4}$

Ans:- $L^{-1}\left\{\frac{s-4}{s^2-4}\right\} = L^{-1}\left\{\frac{s}{s^2-4} - \frac{4}{s^2-4}\right\}$

$= L^{-1}\left\{\frac{s}{s^2-4}\right\} - L^{-1}\left\{\frac{4}{s^2-4}\right\}$

$= L^{-1}\left\{\frac{s}{s^2-4}\right\} - 2L^{-1}\left\{\frac{2}{s^2-4}\right\}$

$= \cosh 2t - 2\sinh 2t$

22. $\frac{60+6s^2+s^4}{s^7}$

Ans:. $L^{-1}\left\{\frac{60+6s^2+s^4}{s^7}\right\}$

$= L^{-1}\left\{\frac{60}{s^7}\right\} + L^{-1}\left\{\frac{6s^2}{s^7}\right\} + L^{-1}\left\{\frac{s^4}{s^7}\right\}$

$= \frac{1}{12}L^{-1}\left\{\frac{6!}{s^{6+1}}\right\} + \frac{1}{4}L^{-1}\left\{\frac{4!}{s^{4+1}}\right\} + \frac{1}{2}L^{-1}\left\{\frac{2!}{s^{2+1}}\right\}$

$= \frac{1}{12}t^6 + \frac{1}{4}t^4 + \frac{1}{2}t^2$ .

23. $\frac{s}{L^2s^2+n^2\pi^2}$

Ans:- $L^{-1}\left\{\frac{s}{L^2s^2+n^2\pi^2}\right\} = \frac{1}{L^2}L^{-1}\left\{\frac{s}{s^2+\frac{n^2\pi^2}{L^2}}\right\} = \frac{1}{L^2}\cos\frac{n\pi}{L}t$ .

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25 $\sum_{k=1}^{5} \frac{a_k}{s+k^2}$

Ans: $L^{-1}\left\{ \sum_{k=1}^{5} \frac{a_k}{s+k^2} \right\} = \sum_{k=1}^{5} a_k L^{-1}\left\{ \frac{1}{s+k^2} \right\} = \sum_{k=1}^{5} a_k e^{-k^2 t}$

26. $\frac{s^4+6s-18}{s^5-3s^4}$

Ans:- $L^{-1}\left\{ \frac{s^4+6s-18}{s^5-3s^4} \right\} = L^{-1}\left\{ \frac{s^4+6(s-3)}{s^4(s-3)} \right\}$

$= L^{-1}\left\{ \frac{s^4}{s^4(s-3)} + \frac{6(s-3)}{s^4(s-3)} \right\}$

$= L^{-1}\left\{ \frac{1}{s-3} + \frac{6}{s^4} \right\}$

$= L^{-1}\left\{ \frac{1}{s-3} \right\} + L^{-1}\left\{ \frac{3!}{s^{3+1}} \right\}$

$= e^{3t} + t^3.$

27. $\frac{1}{(s+\sqrt{2})(s-\sqrt{3})}$

Ans:- We have, by Partial fraction

$\frac{1}{(s+\sqrt{2})(s-\sqrt{3})} = \frac{A}{s+\sqrt{2}} + \frac{B}{s-\sqrt{3}}$

$\Rightarrow \frac{1}{(s+\sqrt{2})(s-\sqrt{3})} = \frac{A(s-\sqrt{3}) + B(s+\sqrt{2})}{(s+\sqrt{2})(s-\sqrt{3})}$

$\Rightarrow 1 = A(s-\sqrt{3}) + B(s+\sqrt{2})$

Put $s = -\sqrt{2}$, $1 = A(-\sqrt{2}-\sqrt{3}) \Rightarrow A = -\frac{1}{(\sqrt{2}+\sqrt{3})}$

Put $s = \sqrt{3}$, $1 = B(\sqrt{2}+\sqrt{3}) \Rightarrow B = \frac{1}{(\sqrt{2}+\sqrt{3})}$

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Thus, $\frac{1}{(s+\sqrt{2})(s-\sqrt{3})} = \frac{-\frac{1}{(\sqrt{2}+\sqrt{3})}}{s+\sqrt{2}} + \frac{\frac{1}{(\sqrt{2}+\sqrt{3})}}{s-\sqrt{3}}$

$$L^{-1}\left\{\frac{1}{(s+\sqrt{2})(s-\sqrt{3})}\right\} = \frac{1}{(\sqrt{2}+\sqrt{3})}\left[L^{-1}\left\{\frac{1}{s-\sqrt{3}}\right\} - L^{-1}\left\{\frac{1}{s+\sqrt{2}}\right\}\right]$$

$$= \frac{1}{(\sqrt{2}+\sqrt{3})}\left[e^{\sqrt{3}t} - e^{-\sqrt{2}t}\right]$$

28. $\frac{2s^3}{s^4-1}$

Ans: $L^{-1}\left\{\frac{2s^3}{s^4-1}\right\} = L^{-1}\left\{\frac{2s^3}{(s^2-1)(s^2+1)}\right\}$

$$= L^{-1}\left\{\frac{s}{s^2-1} + \frac{s}{s^2+1}\right\}$$

$$= L^{-1}\left\{\frac{s}{s^2-1}\right\} + L^{-1}\left\{\frac{s}{s^2+1}\right\}$$

$$= \cosh t + \cos t.$$

29-TO-34

Find the Laplace transform of the following

33. $\sinh t \cos t$

Ans: $L\{\sinh t \cos t\}$

$$= L\left\{\left(\frac{e^t - e^{-t}}{2}\right)\cos t\right\}$$

$$= \frac{1}{2}\left[L\{e^t \cos t\} - L\{e^{-t}\cos t\}\right]$$

$$= \frac{1}{2}\left[\frac{(s-1)}{(s-1)^2+1} - \frac{(s+1)}{(s+1)^2+1}\right]$$

(by First shifting theorem.

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34. $L\{(t+1)^2 e^t\}$

Ans:- Let $f(t) = (t+1)^2$

$$L\{f(t)\} = L\{(t+1)^2\} = L\{t^2+2t+1\}$$

$$= L\{t^2\} + 2L\{t\} + L\{1\}$$

$$= \frac{2}{s^3} + 2\frac{1}{s^2} + \frac{1}{s}$$

$$= \frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s} = F(s)$$

So, by F.S.T

$$L\{(t+1)^2 e^t\} = F(s-1) = \frac{2}{(s-1)^3} + \frac{2}{(s-1)^2} + \frac{1}{s-1}$$

35-TO-40

Find the inverse Laplace transform of the following.

37. $\frac{4}{s^2-2s-3}$

Ans:- We have,

$$\frac{4}{s^2-2s-3} = \frac{4}{(s-3)(s+1)}$$

By Partial fraction

$$\frac{4}{s^2-2s-3} = \frac{4}{(s-3)(s+1)} = \frac{A}{}$$

39. $\frac{s}{(s+\frac{1}{2})^2+1}$

Ans:- $$L^{-1}\left\{\frac{s}{(s+\frac{1}{2})^2+1}\right\} = L^{-1}\left\{\frac{s+\frac{1}{2}-\frac{1}{2}}{(s+\frac{1}{2})^2+1}\right\}$$

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$$= L^{-1}\left\{\frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2+1}\right\} - L^{-1}\left\{\frac{\frac{1}{2}}{(s+\frac{1}{2})^2+1}\right\}$$

$$= L^{-1}\left\{\frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2+1}\right\} - \frac{1}{2}L^{-1}\left\{\frac{1}{(s+\frac{1}{2})^2+1}\right\}$$

$$= e^{-\frac{1}{2}t}\cos t - \frac{1}{2}e^{-\frac{1}{2}t}\sin t \qquad (\text{by } F.S.T)$$

40. $\dfrac{2}{s^2+s+\frac{1}{2}}$

Ans. $$L^{-1}\left\{\frac{2}{s^2+s+\frac{1}{2}}\right\} = L^{-1}\left\{\frac{2}{s^2+2\cdot\frac{1}{2}s+(\frac{1}{2})^2-(\frac{1}{2})^2+\frac{1}{2}}\right\}$$

$$= L^{-1}\left\{\frac{2}{(s+\frac{1}{2})^2+\frac{1}{4}}\right\}$$

$$= 2L^{-1}\left\{\frac{1}{(s+\frac{1}{2})^2+(\frac{1}{2})^2}\right\}$$

$$= 2\times 2\, L^{-1}\left\{\frac{\frac{1}{2}}{(s+\frac{1}{2})^2+(\frac{1}{2})^2}\right\}$$

$$= 4e^{-\frac{1}{2}t}\sin\frac{1}{2}t \qquad (\text{by } F.S.T)$$

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# 5.4 Differentiation and integration of Transforms

## Differentiation Transform Theorem

**Statement:**

Let $L\{f(t)\} = F(s)$

Then $L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$, $n = 1, 2, 3 \cdots\cdots$

**Proof:-**

It is given that,

$$F(s) = L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

Differentiating both sides w.r. to 's'

$$\frac{d}{ds} F(s) = \frac{d}{ds}\left\{\int_0^\infty e^{-st} f(t)\, dt\right\}$$

$$= \int_0^\infty -t\, e^{-st} f(t)\, dt$$

$$= (-1)\int_0^\infty e^{-st}\, t\, f(t)\, dt$$

$$\Rightarrow \int_0^\infty e^{-st}\, t\, f(t)\, dt = (-1)\frac{d}{ds} F(s)$$

$$\Rightarrow L\{t f(t)\} = (-1)^1 \frac{d}{ds} F(s)$$

Now $(-1)^1 \frac{d}{ds} F(s) = L\{t f(t)\} = \int_0^\infty e^{-st}\, t\, f(t)\, dt$

Again diff. both sides w.r. to s

$$(-1)^1 \frac{d^2}{ds^2} F(s) = \frac{d}{ds}\left\{\int_0^\infty e^{-st}\, t\, f(t)\, dt\right\}$$

$$= \int_0^\infty -t\, e^{-st}\, t\, f(t)\, dt$$

$$= (-1)\int_0^\infty e^{-st}\, t^2 f(t)\, dt$$

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$\Rightarrow$ ~~$(-1)^2 = \int_0^\infty e$~~

$$\Rightarrow (-1)^2 \frac{d^2}{ds^2} F(s) = \int_0^\infty e^{-st} t^2 f(t)\,dt$$

$$\Rightarrow (-1)^2 \frac{d^2}{ds^2} F(s) = L\{t^2 f(t)\}$$

$$\Rightarrow L\{t^2 f(t)\} = (-1)^2 \frac{d^2}{ds^2} F(s)$$

Continuing this process $n$ times we can get

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$$

$n = 1, 2 \ldots\ldots$

NOTE:- (I) If any function $f(t)$ is multiplied by possitive integer power of $t$ i.e $t, t^2, t^3, t^4 \cdots$ etc then D.T.T is applicable

(II) Since by using derivative we are calculating the th Laplace transform So it is called D.T.T

Problem Set-5.4 (E. Kreyszig) (8TH EDITION)

1-To-7 Find the Laplace transform

1. $te^t$

Ans:

1st Method (FST)

Let $f(t) = t$, $L\{f(t)\} = L\{t\} = \frac{1}{s^2} = F(s)$

Thus by F.S.T $L\{te^t\} = F(s-1) = \frac{1}{(s-1)^2}$

2nd Method

Let $f(t) = e^t$, $L\{f(t)\} = L\{e^t\} = \frac{1}{s-1} = F(s)$

Thus by D.T.T

$$L\{te^t\} = (-1)^1 \frac{d}{ds}\left(\frac{1}{s-1}\right) = (-1)\left(\frac{-1}{(s-1)^2}\right) = \frac{1}{(s-1)^2}$$

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5. $t\cos\omega t$.

Let $f(t) = t\cos\omega t$

$$L\{f(t)\} = L\{\cos\omega t\} = \frac{s}{s^2+\omega^2} = F(s)$$

Thus, by D.T.T

$$L\{t\cos\omega t\} = (-1)^1 \frac{d}{ds}\left(\frac{s}{s^2+\omega^2}\right)$$

$$= -\left[\frac{(s^2+\omega^2)1 - s\cdot 2s}{(s^2+\omega^2)^2}\right]$$

$$= \frac{s^2-\omega^2}{(s^2+\omega^2)^2}$$

6. $t^2\sin 2t$

Let $f(t) = \sin 2t$, $L\{f(t)\} = L\{\sin 2t\} = \frac{2}{s^2+4} = F(s)$

Thus by D.T.T

$$L\{t^2\sin 2t\} = (-1)^2 \frac{d^2}{ds^2}\left(\frac{2}{s^2+4}\right)$$

$$= 2\frac{d}{ds}\left(\frac{-2s}{(s^2+4)^2}\right)$$

$$= -4\left[\frac{(s^2+4)^2 1 - s\cdot 2(s^2+4)2s}{(s^2+4)^4}\right]$$

$$= -4\left[\frac{s^2+4-4s^2}{(s^2+4)^3}\right]$$

$$= -4\left[\frac{4-3s^2}{(s^2+4)^3}\right]$$

$$= 4\left[\frac{3s^2-4}{(s^2+4)^3}\right]$$

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7. $te^{-t}\sin t$

Let $f(t) = \sin t$, $L\{f(t)\} = L\{\sin t\} = \frac{1}{s^2+1} = F(s)$

Thus, by D.T.T

$$L\{t\sin t\} = (-1)^1 \frac{d}{ds}\left(\frac{1}{s^2+1}\right) = -\left[\frac{-2s}{(s^2+1)^2}\right]$$

$$= \frac{2s}{(s^2+1)^2}$$

By F.S.T

$$L\{e^{-t}\, t\sin t\} = \frac{2(s+1)}{[(s+1)^2+1]^2}$$

## Inverse form of D.T.T

Let $L^{-1}\{F(s)\} = f(t)$

Then $L^{-1}\left\{(-1)^n \frac{d^n}{ds^n} F(s)\right\} = t^n f(t)$, $n = 1, 2, \ldots$

NOTE:- If F(s) is a logarithm function or inverse trigonometric function ($\tan^{-1}$, $\cot^{-1}$) then D.T.T is applicable for finding out the inverse Laplace transform. Because ln and inverse can only be removed by using differentiation.

Problem Set-5.4 E. Kreyszig (8TH EDITION)

13, 14, 16 Find the inverse Laplace transform

13. $\ln \frac{s^2+1}{(s-1)^2}$

Ans:- We have

$$L\{f(t)\} = F(s) = \ln \frac{s^2+1}{(s-1)^2} = \ln(s^2+1) - 2\ln(s-1)$$

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By D.T.T

$$L\{tf(t)\} = (-1)^1 \frac{d}{ds}\left[\ln(s^2+1) - 2\ln(s-1)\right]$$

$$= -\left[\frac{2s}{s^2+1} - 2\frac{1}{s-1}\right]$$

$$\Rightarrow L\{tf(t)\} = \frac{2}{s-1} - \frac{2s}{s^2+1}$$

$$\Rightarrow tf(t) = L^{-1}\left\{\frac{2}{s-1} - \frac{2s}{s^2+1}\right\}$$

$$= 2e^t - 2\cos t$$

$$\Rightarrow f(t) = \frac{2e^t - 2\cos t}{t}$$

Example

Find the inverse Laplace transform of

$$\text{arc cot}(s+1) = \cot^{-1}(s+1)$$

Ans:- We have,

$$L\{f(t)\} = F(s) = \cot^{-1}(s+1)$$

By D.T.T

$$L\{tf(t)\} = (-1)^1 \frac{d}{ds}\left[\cot^{-1}(s+1)\right]$$

$$= -\left[-\frac{1}{(s+1)^2+1}\cdot 1\right]$$

$$= \frac{1}{(s+1)^2+1}$$

$$\Rightarrow tf(t) = L^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}$$

$$= e^{-t}\sin t \quad \text{(by F.S-T)}$$

$$\Rightarrow f(t) = \frac{e^{-t}\sin t}{t}.$$

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# Integration Transform theorem

Let $L\{f(t)\} = F(s)$

Then $L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\,ds$

**Proof**

It is given that

$$F(s) = L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

Integrating both sides w.r.to s from $s = s$ to $s = \infty$

$$\int_s^\infty F(s)\,ds = \int_s^\infty \int_0^\infty e^{-st} f(t)\,dt\,ds$$

$$= \int_0^\infty f(t)\,dt \int_s^\infty e^{-st}\,ds$$

$$= \int_0^\infty f(t)\,dt \left[\frac{e^{-st}}{-t}\right]_s^\infty$$

$$= \int_0^\infty f(t)\,dt \left[0 + \frac{e^{-st}}{t}\right]$$

$$= \int_0^\infty e^{-st} \frac{f(t)}{t}\,dt$$

$$= L\left\{\frac{f(t)}{t}\right\}$$

Thus, $L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\,ds$.

NOTE:- (i) If any function $f(t)$ is divided by $t$ then integration transform theorem is applicable for finding out the Laplace transform.

(ii) Since by means of integration we are calculating the Laplace transform so it is called Integration transform theorem.

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**Example**

Find the Laplace transform of

(i) $\frac{\sin \omega t}{t}$ (ii) $\frac{e^t - 1}{t}$.

**Ans to (i)**

Let $f(t) = \sin \omega t$

$$L\{f(t)\} = L\{\sin \omega t\} = \frac{\omega}{s^2 + \omega^2} = F(s)$$

Thus by integration transform theorem

$$L\left\{\frac{\sin \omega t}{t}\right\} = \int_s^\infty \frac{\omega}{s^2 + \omega^2} ds$$

$$= \omega \cdot \frac{1}{\omega} \left[\tan^{-1} \frac{s}{\omega}\right]_s^\infty$$

$$= \tan^{-1} \infty - \tan^{-1} \frac{s}{\omega}$$

$$= \frac{\pi}{2} - \tan^{-1} \frac{s}{\omega}$$

$$= \cot^{-1} \frac{s}{\omega} \qquad \left(\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}\right)$$

$$= \tan^{-1} \frac{\omega}{s} \qquad \left(\cot^{-1} x = \tan^{-1} \frac{1}{x}\right)$$

**Ans to (ii)**

Let $f(t) = e^t - 1$

$$L\{f(t)\} = L\{e^t - 1\} = L\{e^t\} - L\{1\} = \frac{1}{s-1} - \frac{1}{s} = F(s)$$

Thus, by integration transform theorem

$$L\left\{\frac{e^t - 1}{t}\right\} = \int_s^\infty \left(\frac{1}{s-1} - \frac{1}{s}\right)$$

$$= \left[\ln(s-1) - \ln s\right]_s^\infty$$

$$= \left[\ln \frac{s-1}{s}\right]_s^\infty$$

$$= \left[\ln \left(1 - \frac{1}{s}\right)\right]_s^\infty$$

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$$= \ln 1 - \ln\left(1 - \frac{1}{s}\right)$$

$$= 0 - \ln\left(\frac{s-1}{s}\right)$$

$$= \ln\left(\frac{s-1}{s}\right)^{-1} = \ln\left(\frac{s}{s-1}\right)$$

## Inverse form of Integration transform theorem

Let $L^{-1}\{F(s)\} = f(t)$

Then $L^{-1}\left\{\int_s^\infty F(s)\,ds\right\} = \frac{f(t)}{t}$.

## Problem Set -5.4 (E. Kreyszig) (8TH EDITION)

10, 12, 15

Find the inverse Laplace transform of the following

12. $\frac{2s+6}{(s^2+6s+10)^2}$

Ans:- We have,

$$L\{f(t)\} = F(s) = \frac{2s+6}{(s^2+6s+10)^2}$$

So, by integration transform theorem

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \frac{2s+6}{(s^2+6s+10)^2}\,ds$$

Put $s^2+6s+10 = p$

$\Rightarrow (2s+6)\,ds = dp$

$\int \frac{dp}{p^2} = -\frac{1}{p}$

$$= \left[-\frac{1}{s^2+6s+10}\right]_s^\infty$$

$$= 0 + \frac{1}{s^2+6s+10}$$

$$\Rightarrow \frac{f(t)}{t} = L^{-1}\left\{\frac{1}{s^2+6s+10}\right\}$$

$$= L^{-1}\left\{\frac{1}{s^2+2\cdot 3\cdot s+3^2-3^2+10}\right\}$$

$$= L^{-1}\left\{\frac{1}{(s+3)^2+1}\right\}$$

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$$\Rightarrow \frac{f(t)}{t} = e^{-3t}\sin t \qquad (\text{by F.S.T})$$

$$\Rightarrow f(t) = t\,e^{-3t}\sin t .$$

15. $\dfrac{s}{(s^2+4)^2}$

Ans:- We have,

$$L\{f(t)\} = F(s) = \frac{s}{(s^2+4)^2}$$

So, by integration transform theorem,

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \frac{s}{(s^2+4)^2}\,ds$$

Put $s^2+4 = p$

$2s\,ds = dp$

$$= -\frac{1}{2}\left[\frac{1}{s^2+4}\right]_s^\infty \qquad \int \frac{\frac{1}{2}\,dp}{p^2} = -\frac{1}{2}\,\frac{1}{p}$$

$$= -\frac{1}{2}\left[0 - \frac{1}{s^2+4}\right]$$

$$= \frac{1}{2}\,\frac{1}{s^2+4}$$

$$\Rightarrow \frac{f(t)}{t} = \frac{1}{2}\,L^{-1}\left\{\frac{1}{s^2+4}\right\}$$

$$= \frac{1}{2}\cdot\frac{1}{2}\sin 2t$$

$$\Rightarrow f(t) = \frac{1}{4}\,t\sin 2t .$$

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## 5.6 Partial fractions, Diffrential equations

### Rules of Partial fractions

**Rule-1** (Linear non repeated factors)

$$\frac{f(s)}{(s-a)(s-b)(s-c)} = \frac{A}{s-a} + \frac{B}{s-b} + \frac{C}{s-c}$$

**Rule-2** (Linear repeated factors)

$$\frac{f(s)}{(s-a)^2(s-b)^3} = \frac{A}{s-a} + \frac{B}{(s-a)^2} + \frac{C}{s-b} + \frac{D}{(s-b)^2} + \frac{E}{(s-b)^3}$$

**Rule:-3** (quadratic non repeated factors)

$$\frac{f(s)}{(s^2+a^2)(s^2+b^2)} = \frac{As+B}{s^2+a^2} + \frac{Cs+D}{s^2+b^2}$$

**Rule-4** (quadratic repeated factors)

$$\frac{f(s)}{(s^2+a^2)^2(s^2+b^2)^2} = \frac{As+B}{s^2+a^2} + \frac{Cs+D}{(s^2+a^2)^2} + \frac{Es+F}{s^2+b^2} + \frac{Gs+H}{(s^2+b^2)^2}$$

**Example**

$$\frac{1}{(s+1)(s-2)(s+4)^2(s^2+1)(s^2+3)^2}$$

$$= \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{s+4} + \frac{D}{(s+4)^2} + \frac{Es+F}{s^2+1} + \frac{Gs+H}{s^2+3} + \frac{Is+J}{(s^2+3)^2}$$

**NOTE:-** (Basic rule of Partial fraction)

For partial fraction the degree of denominator must be grater than the degree of numarator. otherwise the partial fraction is not possible.

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Example

$\frac{s^2+1}{s^2-3s+2}$, break it by Partial fraction

Ans:

$$s^2-3s+2 \;\Big|\; \begin{array}{r} s^2+1 \\ s^2-3s+2 \\ \hline 3s-1 \end{array} \;\Big|\; 1$$

Now, we have,

$$\frac{s^2+1}{s^2-3s+2} = 1 + \frac{3s-1}{s^2-3s+2}$$

Thus, by Partial fraction

$$\frac{3s-1}{s^2-3s+2} = \frac{3s-1}{(s-1)(s-2)} = \frac{A}{s-1} + \frac{B}{s-2}$$

$$\Rightarrow 3s-1 = A(s-2) + B(s-1)$$

Put $s=1$, $2 = -A \Rightarrow A = -2$

Put $s=2$, $5 = B \Rightarrow B = 5$

Thus, $\frac{3s-1}{s^2-3s+2} = \frac{-2}{s-1} + \frac{5}{s-2}$

Now, $\frac{s^2+1}{s^2-3s+2} = 1 - \frac{2}{s-1} + \frac{5}{s-2}$.

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# PROBLEM SET - 5.6 (BY E. KREYSZIG) (8TH EDITION)

## NO-1 - TO - 9

Find the inverse Laplace transform of the following.

### 1, 3

3. $\dfrac{s^2+9s-9}{s^3-9s} = \dfrac{s^2+9s-9}{s(s-3)(s+3)}$

We have, by Partial fraction,

$$\frac{s^2+9s-9}{s^3-9s} = \frac{s^2+9s-9}{s(s-3)(s+3)} = \frac{A}{s} + \frac{B}{s-3} + \frac{C}{s+3}$$

$$\Rightarrow s^2+9s-9 = A(s-3)(s+3) + B\,s(s+3) + C\,s(s-3)$$

Put $s=0$, $\quad -9 = -9A \Rightarrow A=1$

Put $s=3$, $\quad 27 = 18B \Rightarrow B = \frac{3}{2}$

Put $s=-3$, $\quad -27 = 18C \Rightarrow C = -\frac{3}{2}$

Thus, $\dfrac{s^2+9s-9}{s^3-9s} = \dfrac{1}{s} + \dfrac{(\frac{3}{2})}{s-3} - \dfrac{(\frac{3}{2})}{s+3}$

$$L^{-1}\left\{\frac{s^2+9s-9}{s^3-9s}\right\} = L^{-1}\left\{\frac{1}{s}\right\} + \frac{3}{2}L^{-1}\left\{\frac{1}{s-3}\right\} - \frac{3}{2}L^{-1}\left\{\frac{1}{s+3}\right\}$$

$$= 1 + \frac{3}{2}e^{3t} - \frac{3}{2}e^{-3t}.$$

### 4, 9

4. $\dfrac{s}{(s+1)^2}$

By Partial fraction

$$\frac{s}{(s+1)^2} = \frac{A}{s+1} + \frac{B}{(s+1)^2}$$

$$\Rightarrow s = A(s+1) + B$$

$$\Rightarrow s = As + (A+B)$$

Equating both sides,

$$A = 1$$

$$A+B=0 \Rightarrow B = -1$$

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Thus, $\frac{s}{(s+1)^2} = \frac{1}{s+1} - \frac{1}{(s+1)^2}$

$$L^{-1}\left\{\frac{s}{(s+1)^2}\right\} = L^{-1}\left\{\frac{1}{s+1}\right\} - L^{-1}\left\{\frac{1}{(s+1)^2}\right\}$$

$$= e^{-t} - te^{-t} \quad \text{(by FST)}$$

2, 5, 7, 8

2. $\frac{s^3+2s^2+2}{s^3(s^2+1)}$

Ans:

$$L^{-1}\left\{\frac{s^3+2s^2+2}{s^3(s^2+1)}\right\} = L^{-1}\left\{\frac{s^3+2(s^2+1)}{s^3(s^2+1)}\right\}$$

$$= L^{-1}\left\{\frac{s^3}{s^3(s^2+1)} + \frac{2(s^2+1)}{s^3(s^2+1)}\right\}$$

$$= L^{-1}\left\{\frac{1}{s^2+1}\right\} + L^{-1}\left\{\frac{2}{s^3}\right\}$$

$$= \sin t + t^2$$

8. $\frac{s^3-7s^2+14s-9}{(s-1)^2(s-2)^3}$

Ans:

$$L^{-1}\left\{\frac{s^3-7s^2+14s-9}{(s-1)^2(s-2)^3}\right\} = L^{-1}\left\{\frac{s^3-6s^2+12s-8-s^2+2s-1}{(s-1)^2(s-2)^3}\right\}$$

$$= L^{-1}\left\{\frac{(s-2)^3-(s-1)^2}{(s-1)^2(s-2)^3}\right\}$$

$$= L^{-1}\left\{\frac{(s-2)^3}{(s-1)^2(s-2)^3}\right\} - L^{-1}\left\{\frac{(s-1)^2}{(s-1)^2(s-2)^3}\right\}$$

$$= L^{-1}\left\{\frac{1}{(s-1)^2}\right\} - L^{-1}\left\{\frac{1}{(s-2)^3}\right\}$$

$$= te^t - \frac{1}{2}t^2e^{2t} \quad \text{(by F.S.T)}$$

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NO-10-TO-13

Prove the following

10. $L^{-1}\left\{\frac{1}{s^4+4a^4}\right\} = \frac{1}{4a^3}(\cosh at \sin at - \sinh at \cos at)$

Ans:- In order to prove this it is sufficient to show,

$$L\left\{\frac{1}{4a^3}(\cosh at \sin at - \sinh at \cos at)\right\} = \frac{1}{s^4+4a^4}$$

LHS

$$\frac{1}{4a^3} L\{\cosh at \sin at - \sinh at \cos at\}$$

$$= \frac{1}{4a^3} L\left\{\left(\frac{e^{at}+e^{-at}}{2}\right)\sin at - \left(\frac{e^{at}-e^{-at}}{2}\right)\cos at\right\}$$

$$= \frac{1}{8a^3}\left[L\{e^{at}\sin at\} + L\{e^{-at}\sin at\} - L\{e^{at}\cos at\} + L\{e^{-at}\cos at\}\right]$$

$$= \frac{1}{8a^3}\left[\frac{a}{(s-a)^2+a^2} + \frac{a}{(s+a)^2+a^2} - \frac{(s-a)}{(s-a)^2+a^2} + \frac{(s+a)}{(s+a)^2+a^2}\right]$$

(by FST)

$$= \frac{1}{8a^3}\left[\frac{2a-s}{s^2+2a^2-2as} + \frac{2a+s}{s^2+2a^2+2as}\right]$$

$$= \frac{1}{8a^3}\left[\frac{(2a-s)(s^2+2a^2+2as) + (2a+s)(s^2+2a^2-2as)}{(s^2+2a^2-2as)(s^2+2a^2+2as)}\right]$$

$$= \frac{1}{8a^3}\left[\frac{2as^2+4a^3+4a^2s-s^3-2a^2s-2as^2 + 2as^2+4a^3-4a^2s+s^3+2a^2s-2as^2}{(s^2+2a^2-2as)(s^2+2a^2+2as)}\right]$$

$$= \frac{1}{8a^3}\left[\frac{8a^3}{(s^2+2a^2)^2-(2as)^2}\right]$$

$$= \frac{1}{s^4+4a^4+4a^2s^2-4a^2s^2}$$

$$= \frac{1}{s^4+4a^4}$$ (Proved)

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## 5.2 Transforms of derivative and integrals, Differential equations.

### Problem Set-5.2 (E. Kreyszig) (8TH EDITION)

**12. (a) to (f)**

Prove the following

(e) $L\{t\cosh at\} = \dfrac{s^2+a^2}{(s^2-a^2)^2}$

Ans:- Let $f(t) = \cosh at$

$$L\{f(t)\} = L\{\cosh at\} = \frac{s}{s^2-a^2} = F(s)$$

Thus, by D.T.T

$$L\{t\cosh at\} = (-1)^1 \frac{d}{ds}\left(\frac{s}{s^2-a^2}\right)$$

$$= -\left[\frac{(s^2-a^2)\,1 - s\cdot 2s}{(s^2-a^2)^2}\right]$$

$$= \frac{s^2-a^2}{(s^2-a^2)^2}$$

(b) $L^{-1}\left\{\dfrac{1}{(s^2+\omega^2)^2}\right\} = \dfrac{1}{2\omega^3}(\sin\omega t - \omega t\cos\omega t)$

Ans:- In order to prove this it is sufficient to show,

$$L\left\{\frac{1}{2\omega^3}(\sin\omega t - \omega t\cos\omega t)\right\} = \frac{1}{(s^2+\omega^2)^2}$$

### Calculation of $L\{t\cos\omega t\}$

Let $f(t) = \cos\omega t$

$$L\{f(t)\} = L\{\cos\omega t\} = \frac{s}{s^2+\omega^2} = F(s)$$

Thus, by DTT

$$L\{t\cos\omega t\} = (-1)^1 \frac{d}{ds}\left(\frac{s}{s^2+\omega^2}\right)$$

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$$= -\left[\frac{(s^2+\omega^2)\,1 - s\cdot 2s}{(s^2+\omega^2)^2}\right]$$

$$= \frac{s^2-\omega^2}{(s^2+\omega^2)^2}$$

LHS

$$\frac{1}{2\omega^3} L\{\sin\omega t - \omega t\cos\omega t\}$$

$$= \frac{1}{2\omega^3}\left[L\{\sin\omega t\} - \omega L\{t\cos\omega t\}\right]$$

$$= \frac{1}{2\omega^3}\left[\frac{\omega}{s^2+\omega^2} - \omega\frac{(s^2-\omega^2)}{(s^2+\omega^2)^2}\right]$$

$$= \frac{1}{2\omega^3}\,\omega\left[\frac{s^2+\omega^2-s^2+\omega^2}{(s^2+\omega^2)^2}\right]$$

$$= \frac{1}{2\omega^2}\left[\frac{2\omega^2}{(s^2+\omega^2)^2}\right]$$

$$= \frac{1}{(s^2+\omega^2)^2} \quad RHS$$

Theorem

Let $L^{-1}\{F(s)\} = f(t)$

Then $L^{-1}\left\{\frac{1}{s}F(s)\right\} = \int_0^t f(\tau)\,d\tau.$

NOTE:- This theorem is applicable for finding out the inverse Laplace transform of $F(s)$ if it is multiplyed by $\frac{1}{s}$.

NO-13 - TO-20

Find the inverse Laplace transform of the following.

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13. $\frac{1}{s^2+4s} = \frac{1}{s(s+4)} = \frac{1}{s} \cdot \frac{1}{s+4}$

Let $F(s) = \frac{1}{s+4}$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s+4}\right\} = e^{-4t} = f(t)$$

Thus, $L^{-1}\left\{\frac{1}{s} \cdot \frac{1}{s+4}\right\} = \int_0^t e^{-4\tau} d\tau = \left[\frac{e^{-4\tau}}{-4}\right]_0^t$

$$= -\frac{1}{4}(e^{-4t} - 1) = \frac{1}{4}(1 - e^{-4t})$$

17. $\frac{1}{s^3 - s} = \frac{1}{s(s^2-1)} = \frac{1}{s} \cdot \frac{1}{s^2-1}$

Let $F(s) = \frac{1}{s^2-1}$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s^2-1}\right\} = \sinh t = f(t)$$

Thus, $L^{-1}\left\{\frac{1}{s} \cdot \frac{1}{s^2-1}\right\} = \int_0^t \sinh\tau \, d\tau = (\cosh\tau)_0^t = \cosh t - 1$

16. $\frac{1}{s^5 + s^3} = \frac{1}{s^3(s^2+1)} = \frac{1}{s} \cdot \frac{1}{s} \cdot \frac{1}{s} \cdot \frac{1}{s^2+1}$

Let $F(s) = \frac{1}{s^2+1}$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t = f(t)$$

$$L^{-1}\left\{\frac{1}{s} \cdot \frac{1}{s^2+1}\right\} = \int_0^t \sin\tau \, d\tau = [-\cos\tau]_0^t = -\cos t + 1$$
$$= 1 - \cos t .$$

$$L^{-1}\left\{\frac{1}{s} \cdot \frac{1}{s} \cdot \frac{1}{s^2+1}\right\} = \int_0^t (1 - \cos\tau) \, d\tau = (\tau - \sin\tau)_0^t = t - \sin t$$

$$L^{-1}\left\{\frac{1}{s} \cdot \frac{1}{s} \cdot \frac{1}{s} \cdot \frac{1}{s^2+1}\right\} = \int_0^t (\tau - \sin\tau) \, d\tau = \left(\frac{\tau^2}{2} + \cos\tau\right)_0^t$$

$$= \frac{t^2}{2} + \cos t - 1 .$$

18. $\frac{1}{s^2}\left(\frac{s-1}{s+1}\right)$

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we have,

$$\frac{1}{s^2}\left(\frac{s-1}{s+1}\right) = \frac{s}{s^2(s+1)} - \frac{1}{s^2(s+1)}$$

$$\Rightarrow \frac{1}{s^2}\left(\frac{s-1}{s+1}\right) = \frac{1}{s(s+1)} - \frac{1}{s^2(s+1)}$$

$$L^{-1}\left\{\frac{1}{s^2}\left(\frac{s-1}{s+1}\right)\right\} = L^{-1}\left\{\frac{1}{s}\cdot\frac{1}{s+1}\right\} - L^{-1}\left\{\frac{1}{s}\cdot\frac{1}{s}\cdot\frac{1}{s+1}\right\} \quad \text{---(1)}$$

Let $F(s) = \frac{1}{s+1}$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t} = f(t)$$

$$L^{-1}\left\{\frac{1}{s}\cdot\frac{1}{s+1}\right\} = \int_0^t e^{-\tau}\,d\tau = \left(-e^{-\tau}\right)_0^t = -e^{-t}+1 = 1-e^{-t}$$

$$L^{-1}\left\{\frac{1}{s}\cdot\frac{1}{s}\cdot\frac{1}{s+1}\right\} = \int_0^t (1-e^{-\tau})\,d\tau = (\tau + e^{-\tau})_0^t$$

$$= t + e^{-t} - 1$$

Putting in equation (1)

$$L^{-1}\left\{\frac{1}{s^2}\left(\frac{s-1}{s+1}\right)\right\} = (1-e^{-t}) - (t+e^{-t}-1)$$

$$= 1 - e^{-t} - t - e^{-t} + 1$$

$$= 2 - 2e^{-t} - t.$$

## Solution of differential equations by using Laplace Transform.

Let $L\{y\} = Y(s)$

STEP:-1

Take the Laplace transform of both sides of the given differential equations and by using the formula

$$L\{y'\} = sL\{y\} - y(0)$$

$$L\{y''\} = s^2L\{y\} - sy(0) - y'(0)$$

Find out the subsidary equation

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STEP:-II

Solution of subsidary equation i.e the value of $Y(s)$

STEP:-III

Solution of the differential equation i.e the value of $y$ by taking the inverse Laplace transform of both sides.

NO-1-TO-9

Solve the following initial value problems.

1. $y' + 3y = 10 \sin t$, $y(0) = 0$

Ans:- we have,

$$y' + 3y = 10 \sin t \quad \text{---- (1)} \qquad y(0) = 0$$

Let $L\{y\} = Y(s)$

Taking the Laplace transform of both sides of equation (1)

$$L\{y'\} + 3L\{y\} = 10\, L\{\sin t\}$$

$$\Rightarrow sL\{y\} - y(0) + 3L\{y\} = 10\, L\{\sin t\}$$

$$\Rightarrow sY(s) - 0 + 3Y(s) = 10 \cdot \frac{1}{s^2+1}$$

$$\Rightarrow (s+3)Y(s) = \frac{10}{s^2+1}$$ which is known as the subsidary equation

$$\text{So,} \quad Y(s) = \frac{10}{(s^2+1)(s+3)} \quad \text{---- (2)}$$

Taking the inverse Laplace transform of both sides

$$L^{-1}\{Y(s)\} = L^{-1}\left\{\frac{10}{(s^2+1)(s+3)}\right\} \quad \text{---- (2)}$$

we have, by Partial fraction.

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$$\frac{10}{(s^2+1)(s+3)} = \frac{As+B}{s^2+1} + \frac{C}{s+3}$$

$$\Rightarrow \quad 10 = (As+B)(s+3) + C(s^2+1)$$

$$\Rightarrow \quad 10 = As^2 + 3As + Bs + 3B + Cs^2 + C$$

$$\Rightarrow \quad 10 = (A+C)s^2 + (3A+B)s + (3B+C)$$

Equating both sides.

$$A + C = 0$$

$$3A + B = 0$$

$$3B + C = 10$$

Solving $\quad A = -1 \quad B = 3 \quad C = 1$

Thus, $\quad \frac{10}{(s^2+1)(s+3)} = \frac{-s+3}{s^2+1} + \frac{1}{s+3}$

Putting in equation (2)

$$L^{-1}\{Y(s)\} = L^{-1}\left\{\frac{-s}{s^2+1} + \frac{3}{s^2+1} + \frac{1}{s+3}\right\}$$

$\Rightarrow y = -\cos t + 3\sin t + e^{-3t}$ is the solution of the given DE.

4. $y'' - y' - 2y = 0$, $\quad y(0) = 8, \quad y'(0) = 7$

Ans:. We have,

$$y'' - y' - 2y = 0 \quad \text{——— (1)}$$

Let $L\{y\} = Y(s)$

Taking the Laplace transform of both sides of equation (1)

$$L\{y''\} - L\{y'\} - 2L\{y\} = 0$$

$$\Rightarrow s^2 L\{y\} - sy(0) - y'(0) - sL\{y\} + y(0) - 2L\{y\} = 0$$

$$\Rightarrow s^2 Y(s) - s \cdot 8 - 7 - sY(s) + 8 - 2Y(s) = 0$$

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$\Rightarrow (s^2 - s - 2)\,Y(s) = 8s - 1$ which is known as the subsidary equation

$$Y(s) = \frac{8s-1}{(s-2)(s+1)}$$

Taking the inverse LT of both sides

$$L^{-1}\{Y(s)\} = L^{-1}\left\{\frac{8s-1}{(s-2)(s+1)}\right\} \quad \text{--- (2)}$$

we have,

$$\frac{8s-1}{(s-2)(s+1)} = \frac{A}{s-2} + \frac{B}{s+1}$$

$$\Rightarrow 8s - 1 = A(s+1) + B(s-2)$$

Put $s = 2$, $15 = 3A \Rightarrow A = 5$

Put $s = -1$, $-9 = -3B \Rightarrow B = 3$

Thus, $$\frac{8s-1}{(s-2)(s+1)} = \frac{5}{s-2} + \frac{3}{s+1}$$

Putting in equation (2)

$$L^{-1}\{Y(s)\} = L^{-1}\left\{\frac{5}{s-2} + \frac{3}{s+1}\right\}$$

$\Rightarrow y = 5e^{2t} - 3e^{-t}$ is the solution of the given DE.

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## Question

Show that

(i) $L\{f'(t)\} = sL\{f(t)\} - f(0)$

(ii) $L\{f''(t)\} = s^2 L\{f(t)\} - sf(0) - f'(0)$

## Ans:-

(i) we have,

$$L\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt$$

$$= \left(e^{-st} f(t)\right)_0^\infty - \int_0^\infty -se^{-st} f(t)\,dt$$

$$= 0 - f(0) + s\int_0^\infty e^{-st} f(t)\,dt$$

$$= -f(0) + sL\{f(t)\}$$

$$= sL\{f(t)\} - f(0)$$

(ii) Since,

$$L\{f'(t)\} = sL\{f(t)\} - f(0)$$

$$\text{&, } L\{f''(t)\} = sL\{f'(t)\} - f'(0)$$

$$= s[sL\{f(t)\} - f(0)] - f'(0)$$

$$= s^2 L\{f(t)\} - sf(0) - f'(0)$$

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9. $y'' + 2y' - 3y = 6e^{-2t}$, $y(0) = 2$, $y'(0) = -14$.

Ans:- We have,

$$y'' + 2y' - 3y = 6e^{-2t} \quad \text{——— (1)}$$

Let $L\{y\} = Y(s)$

Taking the Laplace transform of both sides of equation (1)

$$L\{y''\} + 2L\{y'\} - 3L\{y\} = 6L\{e^{-2t}\}$$

$$\Rightarrow s^2 L\{y\} - sy(0) - y'(0) + 2[sL\{y\} - y(0)] - 3L\{y\} = 6L\{e^{-2t}\}$$

$$\Rightarrow s^2 Y(s) - s \cdot 2 + 14 + 2sY(s) - 2 \cdot 2 - 3Y(s) = 6\frac{1}{s+2}$$

$$\Rightarrow (s^2 + 2s - 3)Y(s) = 2s - 10 + \frac{6}{s+2}$$

$$\Rightarrow (s+3)(s-1)Y(s) = \frac{2s^2 + 4s - 10s - 20 + 6}{s+2}$$

$$\Rightarrow (s+3)(s-1)Y(s) = \frac{2s^2 - 6s - 14}{s+2}$$ which is known as the subsidary equation

$$Y(s) = \frac{2s^2 - 6s - 14}{(s+3)(s-1)(s+2)}$$

Taking the inverse Laplace transform of both sides

$$L^{-1}\{Y(s)\} = L^{-1}\left\{\frac{2s^2 - 6s - 14}{(s+3)(s-1)(s+2)}\right\} \quad \text{——— (2)}$$

By Partial fraction,

$$\frac{2s^2 - 6s - 14}{(s+3)(s-1)(s+2)} = \frac{A}{s+3} + \frac{B}{s-1} + \frac{C}{s+2}$$

$$\Rightarrow 2s^2 - 6s - 14 = A(s-1)(s+2) + B(s+3)(s+2) + C(s+3)(s-1)$$

Put $s = -3$, $22 = 4A \Rightarrow A = \frac{11}{2}$

$s = 1$, $-18 = 12B \Rightarrow B = -\frac{3}{2}$

$s = -2$, $6 = -3C \Rightarrow C = -2$

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Thus, $\frac{2s^2-6s-14}{(s+3)(s-1)(s+2)} = \frac{\frac{11}{2}}{s+3} - \frac{\frac{3}{2}}{s-1} - \frac{2}{s+2}$

Putting in equation (2)

$$L^{-1}\{Y(s)\} = L^{-1}\left\{\frac{\frac{11}{2}}{s+3} - \frac{\frac{3}{2}}{s-1} - \frac{2}{s+2}\right\}$$

$$\Rightarrow y = \frac{11}{2}L^{-1}\left\{\frac{1}{s+3}\right\} - \frac{3}{2}L^{-1}\left\{\frac{1}{s-1}\right\} - 2L^{-1}\left\{\frac{1}{s+2}\right\}$$

$$= \frac{11}{2}e^{-3t} - \frac{3}{2}e^{t} - 2e^{-2t}$$ is the solution of the given DE.

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## 5.7 System of differential equations

### Problem Set-5.7

NO-1-TO-5

Solve the following system of differential equations.

5. $y_1' + y_2 = 2\cos t$ —— (1) $\quad y_1(0) = 0$

$y_1 + y_2' = 0$ —— (2) $\quad y_2(0) = 1$

Ans:-

Let $L\{y_1\} = Y_1(s)$, $L\{y_2\} = Y_2(s)$

Taking the Laplace transform of both sides of eqn (1) & (2)

$$L\{y_1'\} + L\{y_2\} = L\{2\cos t\}$$

$$\Rightarrow sL\{y_1\} - y_1(0) + L\{y_2\} = 2L\{\cos t\}$$

$$\Rightarrow sY_1(s) - 0 + Y_2(s) = 2\frac{s}{s^2+1}$$

$$\Rightarrow sY_1(s) + Y_2(s) = \frac{2s}{s^2+1} \quad \text{—— (3)}$$

$$L\{y_1\} + L\{y_2'\} = 0$$

$$\Rightarrow L\{y_1\} + sL\{y_2\} - y_2(0) = 0$$

$$\Rightarrow Y_1(s) + sY_2(s) - 1 = 0$$

$$\Rightarrow Y_1(s) + sY_2(s) = 1 \quad \text{—— (4)}$$

Solving eqn (1) & (2)

$$s^2Y_1(s) + sY_2(s) = \frac{2s^2}{s^2+1}$$

$$Y_1(s) + sY_2(s) = 1$$

$$(s^2-1)Y_1(s) = \frac{2s^2}{s^2+1} - 1$$

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$$\Rightarrow (s^2-1)\,Y_1(s) = \frac{s^2-1}{s^2+1}$$

$$\Rightarrow Y_1(s) = \frac{1}{s^2+1} \quad \text{——— (5)}$$

Putting in eqn (4)

$$\frac{1}{s^2+1} + s\,Y_2(s) = 1$$

$$\Rightarrow s\,Y_2(s) = 1 - \frac{1}{s^2+1}$$

$$\Rightarrow s\,Y_2(s) = \frac{s^2+1-1}{s^2+1}$$

$$\Rightarrow s\,Y_2(s) = \frac{s^2}{s^2+1}$$

$$\Rightarrow Y_2(s) = \frac{s}{s^2+1} \quad \text{——— (6)}$$

Taking the inverse L.T of both sides of eqn (5) & (6)

$$L^{-1}\{Y_1(s)\} = L^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$\Rightarrow y_1 = \sin t$$

$$L^{-1}\{Y_2(s)\} = L^{-1}\left\{\frac{s}{s^2+1}\right\}$$

$$\Rightarrow y_2 = \cos t$$

Thus, $y_1 = \sin t$, $y_2 = \cos t$ is the soln of the given ~~DOE~~ System of DE

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1. $y_1' = -y_1 + y_2$ ——— (1) $\quad y_1(0) = 1$

$y_2' = -y_1 - y_2$ ——— (2) $\quad y_2(0) = 0$

Ans: Let $L\{y_1\} = Y_1(s) \quad L\{y_2\} = Y_2(s)$

Taking the Laplace transform of both side of eqn (1) & (2)

$$L\{y_1'\} = -L\{y_1\} + L\{y_2\}$$

$$\Rightarrow sL\{y_1\} - y_1(0) = -L\{y_1\} + L\{y_2\}$$

$$\Rightarrow sY_1(s) - 1 = -Y_1(s) + Y_2(s)$$

$$\Rightarrow (s+1)Y_1(s) - Y_2(s) = 1 \quad \text{——— (3)}$$

$$L\{y_2'\} = -L\{y_1\} - L\{y_2\}$$

$$\Rightarrow sL\{y_2\} - y_2(0) = -L\{y_1\} - L\{y_2\}$$

$$\Rightarrow sY_2(s) - 0 = -Y_1(s) - Y_2(s)$$

$$\Rightarrow Y_1(s) + (s+1)Y_2(s) = 0 \quad \text{——— (4)}$$

Solving eqn (3) & (4)

$$(s+1)^2 Y_1(s) - (s+1)Y_2(s) = s+1$$

$$Y_1(s) + (s+1)Y_2(s) = 0$$

---

$$[(s+1)^2 + 1]\, Y_1(s) = s+1$$

$$\Rightarrow Y_1(s) = \frac{s+1}{(s+1)^2+1} \quad \text{——— (5)}$$

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From eq$^{n}$ (4)

$$(s+1)X_2(s) = -Y_1(s) = -\frac{(s+1)}{(s+1)^2+1}$$

$$\Rightarrow X_2(s) = -\frac{1}{(s+1)^2+1} \quad \text{---(6)}$$

Taking the inverse LT of both sides of eq$^{n}$ (5) & (6)

$$L^{-1}\{Y_1(s)\} = L^{-1}\left\{\frac{s+1}{(s+1)^2+1}\right\}$$

$$\Rightarrow y_1 = e^{-t}\cos t \quad \text{(by F.S-D)}$$

$$L^{-1}\{X_2(s)\} = L^{-1}\left\{-\frac{1}{(s+1)^2+1}\right\}$$

$$\Rightarrow z = -e^{-t}\sin t$$

Soln, $y_1 = e^{-t}\cos t$, $z_2 = -e^{-t}\sin t$ is the soln of the given system of DE.

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## Laplace transform of periodic functions

Let $f(t)$ be a periodic function with period $P > 0$ so,

$$L\{f(t)\} = \frac{1}{1-e^{-Ps}} \int_0^P e^{-st} f(t)\, dt$$

**Q:** Find the Laplace transform of the following periodic functions.

(i) $f(t) = t, \quad 0 < t < 2\pi$

(ii) $f(t) = \begin{cases} -1, & 0 < t < \pi \\ 0, & \pi < t < 2\pi \end{cases}$

**Ans to (ii)**

$$L\{f(t)\} = \frac{1}{1-e^{-2\pi s}} \int_0^{2\pi} e^{-st} f(t)\, dt$$

$$= \frac{e^{2\pi s}}{e^{2\pi s}-1} \left[ t\frac{e^{-st}}{-s} - \frac{e^{-st}}{s^2} \right]_0^{2\pi}$$

$$= \frac{e^{2\pi s}}{e^{2\pi s}-1} \left[ \frac{2\pi e^{-2\pi s}}{-s} - \frac{e^{-2\pi s}}{s^2} - 0 + \frac{1}{s^2} \right]$$

$$= \frac{e^{2\pi s}}{e^{2\pi s}-1} \left[ -\frac{2\pi}{s} e^{-2\pi s} + \frac{1}{s^2}(1-e^{-2\pi s}) \right]$$

$$= \frac{1}{e^{2\pi s}-1} \left[ -\frac{2\pi}{s} + \frac{1}{s^2}(e^{2\pi s}-1) \right]$$

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(11) $$L\{f(t)\} = \frac{1}{1-e^{-2\pi s}} \int_0^{2\pi} e^{-st} f(t)\,dt$$

$$= \frac{e^{2\pi s}}{e^{2\pi s}-1}\left[\int_0^{\pi} e^{-st} f(t)\,dt + \int_{\pi}^{2\pi} e^{-st} f(t)\,dt\right]$$

$$= \frac{e^{2\pi s}}{e^{2\pi s}-1}\left[\int_0^{\pi} e^{-st}(-1)\,dt + 0\right]$$

$$= \frac{e^{2\pi s}}{e^{2\pi s}-1}\left[-\frac{e^{-st}}{-s}\right]_0^{\pi}$$

$$= \frac{e^{2\pi s}}{s(e^{2\pi s}-1)}\left[e^{-\pi s}-1\right]$$

$$= \frac{e^{\pi s}-e^{-2\pi s}}{s(e^{2\pi s}-1)}$$

$$= \frac{e^{\pi s}(1-e^{\pi s})}{s(e^{2\pi s}-1)}$$

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## 5.5 Convolution. Integral equations

### Convolution Theorem

Let $H(s)$ be the product of two functions $F(s)$ and $G(s)$. That is $H(s) = F(s)\,G(s)$

Let $L^{-1}\{F(s)\} = f(t)$

$L^{-1}\{G(s)\} = g(t)$

Then by Convolution Theorem,

$$h(t) = L^{-1}\{H(s)\} = L^{-1}\{F(s)\,G(s)\} = f * g$$

$$= \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

### Properties of Convolution

(i) Commutative $\quad f * g = g * f$

(ii) Distributive $\quad f * (g_1 + g_2) = f * g_1 + f * g_2$

(iii) Associative $\quad (f * g) * v = f * (g * v)$

### NO-1-TO-9

Calculate the following convolutions

3. $e^t * e^{-t}$

$$= \int_0^t e^{\tau}\, e^{-(t-\tau)}\, d\tau$$

$$= \int_0^t e^{\tau}\, e^{-t+\tau}\, d\tau$$

$$= \int_0^t e^{\tau}\, e^{-t}\, e^{\tau}\, d\tau$$

$$= e^{-t} \int_0^t e^{2\tau}\, d\tau$$

$$= e^{-t} \left[\frac{e^{2\tau}}{2}\right]_0^t$$

$$= \frac{1}{2} e^{-t} \left[e^{2t} - 1\right] = \frac{1}{2}\left(e^t - e^{-t}\right)$$

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5. $\sin \omega t * \cos \omega t$

$$= \int_0^t \sin \omega\tau \cos \omega(t-\tau)\, d\tau$$

$$= \frac{1}{2} \int_0^t 2 \sin \omega\tau \cos(\omega t - \omega\tau)\, d\tau$$

$$= \frac{1}{2} \int_0^t [\sin(\omega\tau + \omega t - \omega\tau) + \sin(\omega\tau - \omega t + \omega\tau)]\, d\tau$$

$$= \frac{1}{2} \int_0^t [\sin \omega t + \sin(2\omega\tau - \omega t)]\, d\tau$$

$$= \frac{1}{2} \left[ \tau \sin \omega t - \frac{1}{2\omega} \cos(2\omega\tau - \omega t) \right]_0^t$$

$$= \frac{1}{2} \left[ t \sin \omega t - \frac{1}{2\omega} \cos \omega t - 0 + \frac{1}{2\omega} \cos \omega t \right]$$

$$= \frac{1}{2} t \sin \omega t .$$

NO-10-TO-18

Find the inverse Laplace transform by using convolution theorem.

11. $\frac{1}{s^2(s-1)}$

Ans: Let $\frac{1}{s^2(s-1)} = \frac{1}{s^2} \cdot \frac{1}{s-1}$

Let $F(s) = \frac{1}{s^2}$ $\quad G(s) = \frac{1}{s-1}$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s^2}\right\} = t = f(t)$$

$$L^{-1}\{G(s)\} = L^{-1}\left\{\frac{1}{s-1}\right\} = e^t = g(t)$$

Thus, by convolution theorem,

$$L^{-1}\left\{\frac{1}{s^2(s-1)}\right\} = t * e^t = \int_0^t \tau e^{t-\tau} d\tau$$

$$= \int_0^t \tau e^t e^{-\tau} d\tau$$

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$$= e^t \int_0^t \gamma e^{-\gamma} d\gamma$$

$$= e^t \left[ -\gamma e^{-\gamma} - e^{-\gamma} \right]_0^t$$

$$= e^t \left[ -t e^{-t} - e^{-t} - 0 + 1 \right]$$

$$= -t - 1 + e^t = e^t - t - 1 .$$

14. $\dfrac{s^2}{(s^2+\omega^2)^2}$

Let $\dfrac{s^2}{(s^2+\omega^2)^2} = \dfrac{s}{s^2+\omega^2} \cdot \dfrac{s}{s^2+\omega^2}$

Let $F(s) = \dfrac{s}{s^2+\omega^2}$ $\quad G(s) = \dfrac{s}{s^2+\omega^2}$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{s}{s^2+\omega^2}\right\} = \cos\omega t = f(t)$$

$$L^{-1}\{G(s)\} = L^{-1}\left\{\frac{s}{s^2+\omega^2}\right\} = \cos\omega t = g(t)$$

By convolution theorem,

$$L^{-1}\left\{\frac{s^2}{(s^2+\omega^2)^2}\right\} = \cos\omega t * \cos\omega t$$

$$= \int_0^t \cos\omega\gamma \cos\omega(t-\gamma)\, d\gamma$$

$$= \frac{1}{2}\int_0^t 2\cos\omega\gamma \cos(\omega t - \omega\gamma)\, d\gamma$$

$$= \frac{1}{2}\int_0^t \left[\cos(\omega\gamma + \omega t - \omega\gamma) + \cos(\omega\gamma - \omega t + \omega\gamma)\right] d\gamma$$

$$= \frac{1}{2}\int_0^t \left[\cos\omega t + \cos(2\omega\gamma - \omega t)\right] d\gamma$$

$$= \frac{1}{2}\left[\gamma\cos\omega t + \frac{1}{2\omega}\sin(2\omega\gamma - \omega t)\right]_0^t$$

$$= \frac{1}{2}\left[t\cos\omega t + \frac{1}{2\omega}\sin\omega t - 0 + \frac{1}{2\omega}\sin\omega t\right]$$

$$= \frac{1}{2}\left[t\cos\omega t + \frac{1}{\omega}\sin\omega t\right]$$

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17. $\frac{\omega}{s^2(s^2+\omega^2)}$

Let, $\frac{\omega}{s^2(s^2+\omega^2)} = \frac{\omega}{s^2+\omega^2} \cdot \frac{1}{s^2}$

Let $F(s) = \frac{\omega}{s^2+\omega^2}$ $\quad G(s) = \frac{1}{s^2}$

$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{\omega}{s^2+\omega^2}\right\} = \sin\omega t = f(t)$

$L^{-1}\{G(s)\} = L^{-1}\left\{\frac{1}{s^2}\right\} = t = g(t)$

By convolution theorem,

$$L^{-1}\left\{\frac{\omega}{s^2(s^2+\omega^2)}\right\} = \sin\omega t * t = \int_0^t \sin\omega\tau \cdot (t-\tau)\,d\tau$$

$$= \left[-(t-\tau)\frac{\cos\omega\tau}{\omega} - \frac{\sin\omega\tau}{\omega^2}\right]_0^t$$

$$= \left[0 - \frac{\sin\omega t}{\omega^2} + \frac{t}{\omega} + 0\right]$$

$$= \frac{t}{\omega} - \frac{\sin\omega t}{\omega^2}$$

19, 20 Solve the following initial value problems by using convolution theorem.

20. $y'' + y = t$ ——— (1) $\quad y(0)=0,\ y'(0)=0$

Ans:- Let $L\{y\} = Y(s)$

Taking the Laplace transform of both sides of equation (1)

$L\{y''\} + L\{y\} = L\{t\}$

$\Rightarrow s^2 L\{y\} - s\,y(0) - y'(0) + L\{y\} = L\{t\}$

$\Rightarrow s^2 Y(s) - s\cdot 0 - 0 + Y(s) = \frac{1}{s^2}$

$\Rightarrow (s^2+1)\,Y(s) = \frac{1}{s^2}$ which is known as the subsidary equation

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$$\Rightarrow Y(s) = \frac{1}{(s^2+1)\,s^2}$$

Taking the inverse Laplace transform of both sides,

$$L^{-1}\{Y(s)\} = L^{-1}\left\{\frac{1}{(s^2+1)\,s^2}\right\} \quad \text{——(2)}$$

Calculation of $L^{-1}\left\{\frac{1}{(s^2+1)\,s^2}\right\}$

Let $\frac{1}{(s^2+1)\,s^2} = \frac{1}{s^2+1}\cdot\frac{1}{s^2}$

Let $F(s) = \frac{1}{s^2+1}$ $\quad G(s) = \frac{1}{s^2}$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t = f(t)$$

$$L^{-1}\{G(s)\} = L^{-1}\left\{\frac{1}{s^2}\right\} = t = g(t)$$

By convolution theorem

$$L^{-1}\left\{\frac{1}{(s^2+1)\,s^2}\right\} = \sin t * t = \int_0^t \sin\tau\,(t-\tau)\,d\tau$$

$$= \left[-(t-\tau)\cos\tau - \sin\tau\right]_0^t$$

$$= 0 - \sin t + t + 0$$

$$= t - \sin t$$

Putting in equation (2)

$y = t - \sin t$ is the solution of the given DE.

27-TO-33

Solve the following integral equations

30. $y(t) = \sin 2t + \int_0^t y(\tau)\sin 2(t-\tau)\,d\tau$

Ans... Now $y(t) = \sin 2t + y(t) * \sin 2t$ ——(1)

Let $L\{y(t)\} = Y(s)$

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Taking the Laplace transform of both sides of equation (1)

$$L\{y(t)\} = L\{\sin 2t\} + L\{y(t) * \sin 2t\}$$

$$\Rightarrow Y(s) = \frac{2}{s^2+4} + Y(s)\frac{2}{s^2+4}$$

$$\Rightarrow \left(1 - \frac{2}{s^2+4}\right) Y(s) = \frac{2}{s^2+4}$$

$$\Rightarrow \frac{s^2+2}{s^2+4} Y(s) = \frac{2}{s^2+4}$$

$$\Rightarrow Y(s) = \frac{2}{s^2+2}$$

Taking the inverse Laplace transform of both sides

$$L^{-1}\{Y(s)\} = L^{-1}\left\{\frac{2}{s^2+2}\right\}$$

$$\Rightarrow y(t) = \sqrt{2}\, L^{-1}\left\{\frac{\sqrt{2}}{s^2+(\sqrt{2})^2}\right\}$$

$$\Rightarrow y(t) = \sqrt{2} \sin \sqrt{2}t$$ is the solution of the given integral equation.

31. $y(t) = te^t - 2e^t \int_0^t e^{-\tau} y(\tau)\, d\tau$

Ans. we have,

$$y(t) = te^t - 2\int_0^t e^{t-\tau} y(\tau)\, d\tau$$

$$\Rightarrow y(t) = te^t - 2\int_0^t y(\tau)\, e^{t-\tau}\, d\tau$$

$$\Rightarrow y(t) = te^t - 2\, y(t) * e^t \quad \text{——— (1)}$$

Let $L\{y(t)\} = Y(s)$

Taking the Laplace transform of both sides of eqn (1)

$$L\{y(t)\} = L\{te^t\} - 2\, L\{y(t) * e^t\}$$

$$\Rightarrow Y(s) = \frac{1}{(s-1)^2} - 2\, Y(s)\frac{1}{s-1} \quad \text{(by F-S-T)}$$

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$$\Rightarrow \left(1+\frac{2}{s-1}\right) Y(s) = \frac{1}{(s-1)^2}$$

$$\Rightarrow \frac{s+1}{s-1} Y(s) = \frac{1}{(s-1)^2}$$

$$\Rightarrow (s+1) Y(s) = \frac{1}{(s-1)}$$

$$\Rightarrow Y(s) = \frac{1}{s^2-1}$$

Taking the inverse Laplace transform of both sides

$$L^{-1}\{Y(s)\} = L^{-1}\left\{\frac{1}{s^2-1}\right\}$$

$\Rightarrow y(t) = \sinh t$ is the solution of the given integral equation.

Q:- Show that convolution is distributive i.e

$$f * (g_1 + g_2) = f * g_1 + f * g_2$$

Ans:-

$$f * (g_1 + g_2)$$

$$= \int_0^t f(\tau)\,(g_1+g_2)(t-\tau)\,d\tau$$

$$= \int_0^t f(\tau)\,[g_1(t-\tau) + g_2(t-\tau)]\,d\tau$$

$$= \int_0^t f(\tau)\, g_1(t-\tau)\,d\tau + \int_0^t f(\tau)\, g_2(t-\tau)\,d\tau$$

$$= f * g_1 + f * g_2 .$$

Q:- Show that convolution is commutative

$$f * g = g * f$$

Ans: we have,

$$f * g = \int_0^t f(\tau)\, g(t-\tau)\,d\tau$$

put $t-\tau = v$ when $\tau = 0,\ v = t$

$\tau = t,\ v = 0$

$\Rightarrow -d\tau = dv$

$\Rightarrow d\tau = -dv$

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Thus. $f * g = \int_t^0 f(t-v)\, g(v) \cdot -dv$

$= \int_0^t g(v)\, f(t-v)\, dv$

$= g * f$ .

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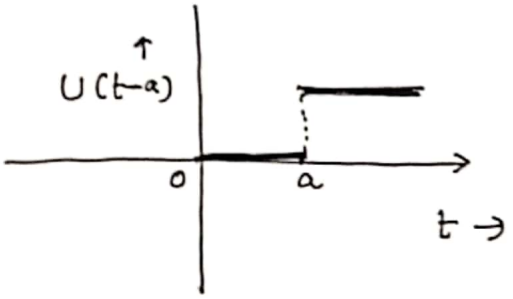
## 5.3 Unit Step Function, second Shifting Theorem, Dirac's Delta Function

### Unit step Function

We have the unit step function is denoted by $U(t-a)$ and it is defined as

$$U(t-a) = \begin{cases} 0, & t < a \\ 1, & t > a \end{cases}$$

Graph




Question

Find the Laplace transform of $U(t-a)$

Ans:- we have,

$$U(t-a) = \begin{cases} 0, & t < a \\ 1, & t > a \end{cases}$$

$$L\{U(t-a)\} = \int_0^\infty e^{-st}\, U(t-a)\, dt$$

$$= \int_0^a e^{-st}\, U(t-a)\, dt + \int_a^\infty e^{-st}\, U(t-a)\, dt$$

$$= 0 + \int_a^\infty e^{-st}\, 1\, dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_a^\infty$$

$$= -\frac{1}{s}\left[0 - e^{-as}\right]$$

$$= \frac{e^{-as}}{s}.$$

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Thus. $\boxed{L\{U(t-a)\} = \frac{e^{-as}}{s}}$

Example

$$L\{U(t-1)\} = \frac{e^{-s}}{s}$$

$$L\{U(t-2)\} = \frac{e^{-2s}}{s}$$

$$L\{U(t-3)\} = \frac{e^{-3s}}{s}.$$

Inverse form

$$L^{-1}\left\{\frac{e^{-as}}{s}\right\} = U(t-a)$$

Ex:-

$$L^{-1}\left\{\frac{e^{-4s}}{s}\right\} = U(t-4)$$

$$L^{-1}\left\{\frac{e^{-3s}}{s}\right\} = U(t-3)$$

Dirac's Delta function

We have the diracs delta function is denoted by $\delta(t-a)$ and it is defined as

$$\delta(t-a) = \begin{cases} \infty, & \text{if } t=a \\ 0, & \text{otherwise} \end{cases}$$

NOTE:-

$$L\{\delta(t-a)\} = e^{-as}$$

Ex:-

$$L\{\delta(t-1)\} = e^{-s}$$

$$L\{\delta(t-2)\} = e^{-2s}$$

$$L\{\delta(t)\} = 1$$

Inverse form $L^{-1}\{e^{-as}\} = \delta(t-a)$

Ex: $L^{-1}\{1\} = \delta(t)$ $\qquad L^{-1}\{e^{-3s}\} = \delta(t-3)$

$L^{-1}\{e^{-2s}\} = \delta(t-2)$

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## Second shifting theorem

Let $L\{f(t)\} = F(s)$

Then

$$L\{f(t-a)\cup(t-a)\} = e^{-as}F(s)$$

**Proof:-**

It is given that, $L\{f(t)\} = F(s)$

Now,

$$L\{f(t-a)\cup(t-a)\} = \int_0^\infty e^{-st} f(t-a)\cup(t-a)\,dt$$

$$= \int_0^a e^{-st} f(t-a)\cup(t-a)\,dt + \int_a^\infty e^{-st} f(t-a)\cup(t-a)\,dt$$

$$= 0 + \int_a^\infty e^{-st} f(t-a)\cdot 1\,dt$$

$$= \int_a^\infty e^{-st} f(t-a)\,dt$$

Put $t-a = v$

$dt = dv$

when, $t = a$, $v = 0$

$t = \infty$, $v = \infty$

Thus, $L\{f(t-a)\cup(t-a)\} = \int_0^\infty e^{-s(a+v)} f(v)\,dv$

$$= \int_0^\infty e^{-as}\cdot e^{-sv} f(v)\,dv$$

$$= e^{-as}\int_0^\infty e^{-sv} f(v)\,dv$$

$$= e^{-as} L\{f(t)\}$$

$$= e^{-as} F(s)$$

(Proved)

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## Example

Find the Laplace transform of

(i) $(t-2)^3 \cup(t-2)$ (ii) $e^{t-1} \cup(t-1)$

(iii) $\sin(t-\pi) \cup(t-\pi)$ (iv) $\cos(t-2\pi) \cup(t-2\pi)$

Ans:-

(i) Here $a=2$, $f(t-2) = (t-2)^3$

$$f(t) = t^3$$

$$L\{f(t)\} = L\{t^3\} = \frac{6}{s^4} = F(s)$$

Thus by S.S.T

$$L\{(t-2)^3 \cup(t-2)\} = e^{-2s} \frac{6}{s^4} = \frac{6e^{-2s}}{s^4}$$

(ii) Here $a=1$, $f(t-1) = e^{t-1}$

$$f(t) = e^t$$

$$L\{f(t)\} = L\{e^t\} = \frac{1}{s-1} = F(s)$$

Thus by S.S.T

$$L\{e^{t-1} \cup(t-1)\} = e^{-1s} \frac{1}{s-1} = \frac{e^{-s}}{s-1}$$

(iii) Here $a = \pi$, $f(t-\pi) = \sin(t-\pi)$

$$f(t) = \sin t$$

$$L\{f(t)\} = L\{\sin t\} = \frac{1}{s^2+1} = F(s)$$

Thus, by S.S.T

$$L\{\sin(t-\pi) \cup(t-\pi)\} = e^{-\pi s} \frac{1}{s^2+1} = \frac{e^{-\pi s}}{s^2+1}$$

(iv) Here $a = 2\pi$, $f(t-2\pi) = \cos(t-2\pi)$

$$f(t) = \cos t$$

$$L\{f(t)\} = L\{\cos t\} = \frac{s}{s^2+1} = F(s)$$

Thus, by S.S.T

$$L\{\cos(t-2\pi) \cup(t-2\pi)\} = e^{-2\pi s} \frac{s}{s^2+1} = \frac{s e^{-2\pi s}}{s^2+1}$$

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## Example

Find the Laplace transform of

(i) $t^2 u(t-3)$

(ii) $e^{2t} u(t-1)$

(iii) $\sin t\, u(t-\pi)$

(iv) $\cos t\, u(t-2\pi)$

Ans:-

(i) $L\{t^2 u(t-3)\}$

$= L\{(t-3+3)^2 u(t-3)\}$

$= L\{(t-3)^2 u(t-3)\} + 6L\{(t-3)u(t-3)\} + 9L\{u(t-3)\}$

$= \frac{2}{s^3} e^{-3s} + 6\frac{1}{s^2} e^{-3s} + 9\frac{e^{-3s}}{s}$ (by S.S.T)

$= \left(\frac{2}{s^3} + \frac{6}{s^2} + 9\right) e^{-3s}$

(ii) $L\{e^{2t} u(t-1)\}$

$= L\{e^{2(t-1+1)} u(t-1)\}$

$= L\{e^{2(t-1)} e^2 u(t-1)\}$

$= e^2 L\{e^{2(t-1)} u(t-1)\}$

$= e^2 \frac{1}{s-2} e^{-s} = \frac{e^{2-s}}{s-2}$ (by S.S.T)

(iii) $L\{\sin t\, u(t-\pi)\}$

$= -L\{\sin(t-\pi) u(t-\pi)\}$

$= -\frac{1}{s^2+1} e^{-\pi s}$ (by S.S.T)

(iv) $L\{\cos t\, u(t-2\pi)\}$

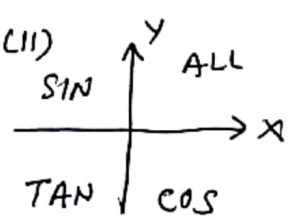
$= L\{\cos(t-2\pi) u(t-2\pi)\}$

$= \frac{s}{s^2+1} e^{-2\pi s}$ (by S.S.T)

(i) $\sin(-\theta) = -\sin\theta$

$\cos(-\theta) = \cos\theta$

(ii)

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**Example:-**

Find the Laplace transform of the following

(i) $t^2$ $(0<t<1)$ (ii) $\sin \omega t$ $(0<t<\pi/\omega)$ (iii) $e^t$ $(0<t<1)$

**Ans:-**

(i) In unit step function $f(t)$ can be written as

$$f(t) = t^2 \cup(t-0) - t^2 \cup(t-1)$$

$$= (t-0)^2 \cup(t-0) - (t-1+1)^2 \cup(t-1)$$

$$= (t-0)^2 \cup(t-0) - (t-1)^2 \cup(t-1) - 2(t-1)\cup(t-1) - \cup(t-1)$$

$$L\{f(t)\} = L\{(t-0)^2 \cup(t-0)\} - L\{(t-1)^2 \cup(t-1)\} - 2L\{(t-1)\cup(t-1)\} - L\{\cup(t-1)\}$$

$$= \frac{2}{s^3} e^{-0s} - \frac{2}{s^3} e^{-1s} - 2\frac{1}{s^2} e^{-1s} - \frac{e^{-s}}{s}$$

(by S.S.T)

$$= \frac{2}{s^3} - \frac{2}{s^3} e^{-s} - \frac{2}{s^2} e^{-s} - \frac{e^{-s}}{s}$$

(ii) In unit step function $f(t)$ can be written as

$$f(t) = \sin\omega t \cup(t-0) - \sin\omega t \cup(t-\tfrac{\pi}{\omega})$$

$$= \sin\omega(t-0)\cup(t-0) + \sin\omega(t-\tfrac{\pi}{\omega})\cup(t-\tfrac{\pi}{\omega})$$

$$L\{f(t)\} = L\{\sin\omega(t-0)\cup(t-0)\} + L\{\sin\omega(t-\tfrac{\pi}{\omega})\cup(t-\tfrac{\pi}{\omega})\}$$

$$= \frac{\omega}{s^2+\omega^2} e^{-0s} + \frac{\omega}{s^2+\omega^2} e^{-\frac{\pi}{\omega}s}$$

$$= \frac{\omega}{s^2+\omega^2}\left(1 + e^{-\frac{\pi}{\omega}s}\right)$$ (by S.S.T)

(iii) In unit step function $f(t)$ can be written as

$$f(t) = e^t \cup(t-0) - e^t \cup(t-1)$$

$$= e^{t-0} \cup(t-0) - e^{(t-1+1)} \cup(t-1)$$

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$$\Rightarrow f(t) = e^{t-0}\cup(t-0) - e^{(t-1)}e\cup(t-1)$$

$$L\{f(t)\} = L\{e^{t-0}\cup(t-0)\} - eL\{e^{t-1}\cup(t-1)\}$$

$$= \frac{1}{s-1}e^{-0s} - e\frac{1}{s-1}e^{-1s}$$

$$= \frac{1}{s-1}(1-e^{1-s}) \quad (\text{by } S.S.T)$$

## Inverse form of second shifting theorem

Let $L^{-1}\{F(s)\} = f(t)$

Then $L^{-1}\{e^{-as}F(s)\} = f(t-a)\cup(t-a)$

**Example:**

Find the inverse Laplace transform of

(i) $\frac{e^{-3s}}{s^3}$ (ii) $\frac{e^{-2\pi s}}{s^2+2s+2}$ (iii) $\frac{s\,e^{-2s}}{s^2+\pi^2}$

**Ans to (i)**

Here $a=3$, $F(s) = \frac{1}{s^3}$, $L^{-1}\{F(s)\} = L^{-1}\{\frac{1}{s^3}\} = \frac{1}{2}t^2 = f(t)$

Thus, by S.S.T

$$L^{-1}\left\{\frac{e^{-3s}}{s^3}\right\} = f(t-3)\cup(t-3) = \frac{1}{2}(t-3)^2\cup(t-3)$$

(ii) Here $a=2\pi$, $F(s) = \frac{1}{s^2+2s+2}$, $L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s^2+2s+2}\right\}$

$$= L^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}$$

$$= e^{-t}\sin t \quad (\text{by F.S.T})$$

Thus, by S.S.T

$$L^{-1}\left\{\frac{e^{-2\pi s}}{s^2+2s+2}\right\} = f(t-2\pi)\cup(t-2\pi) = e^{-(t-2\pi)}\sin(t-2\pi)\cup(t-2\pi)$$

(iii) Here $a=2$, $F(s) = \frac{s}{s^2+\pi^2}$, $L^{-1}\{F(s)\} = L^{-1}\left\{\frac{s}{s^2+\pi^2}\right\} = \cos\pi t = f(t)$

Thus, by S.S.T

$$L^{-1}\left\{\frac{s\,e^{-2s}}{s^2+\pi^2}\right\} = f(t-2)\cup(t-2) = \cos\pi(t-2)\cup(t-2)$$

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Example:-1

Find the inverse Laplace transform of

$$\frac{s^3-3s^2+6s-4}{(s^2-2s+2)^2}$$

Ans:- We have by partial fraction

$$\frac{s^3-3s^2+6s-4}{(s^2-2s+2)^2} = \frac{As+B}{s^2-2s+2} + \frac{Cs+D}{(s^2-2s+2)^2}$$

$$\Rightarrow s^3-3s^2+6s-4 = (As+B)(s^2-2s+2) + Cs+D$$

$$\Rightarrow s^3-3s^2+6s-4 = As^3-2As^2+2As+Bs^2-2Bs+2B + Cs+D$$

$$\Rightarrow s^3-3s^2+6s-4 = As^3+(-2A+B)s^2+(2A-2B+C)s + (2B+D)$$

Equating both sides,

$$A = 1$$

$$-2A+B = -3 \Rightarrow B = -1$$

$$2A-2B+C = 6 \Rightarrow C = 2$$

$$2B+D = -4 \Rightarrow D = -2$$

Thus, $$\frac{s^3-3s^2+6s-4}{(s^2-2s+2)^2} = \frac{s-1}{s^2-2s+2} + \frac{2s-2}{(s^2-2s+2)^2}$$

$$L^{-1}\left\{\frac{s^3-3s^2+6s-4}{(s^2-2s+2)^2}\right\} = L^{-1}\left\{\frac{s-1}{s^2-2s+2}\right\} + L^{-1}\left\{\frac{2s-2}{(s^2-2s+2)^2}\right\} \quad \text{---(1)}$$

Now $$L^{-1}\left\{\frac{s-1}{s^2-2s+2}\right\} = L^{-1}\left\{\frac{s-1}{(s-1)^2+1}\right\}$$

$$= e^t \cos t \quad \text{(by F.S.T)}$$

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$$L^{-1}\left\{\frac{2s-2}{(s^2-2s+2)^2}\right\}$$

---

we have, $L\{f(t)\} = F(s) = \dfrac{2s-2}{(s^2-2s+2)^2}$

Thus, by I.T.T $L\left\{\dfrac{f(t)}{t}\right\} = \displaystyle\int_s^\infty \frac{2s-2}{(s^2-2s+2)^2}\,ds$ Put $s^2-2s+2 = p$

$\Rightarrow (2s-2)\,ds = dp$

$\int \frac{dp}{p^2} = -\frac{1}{p}$

$$= -\left[\frac{1}{s^2-2s+2}\right]_s^\infty$$

$$= -\left[0 - \frac{1}{s^2-2s+2}\right]$$

$$= \frac{1}{(s-1)^2+1}$$

$$\Rightarrow \frac{f(t)}{t} = L^{-1}\left\{\frac{1}{(s-1)^2+1}\right\}$$

$$= e^t \sin t \qquad \text{(by F.S.T)}$$

$$\Rightarrow f(t) = te^t \sin t$$

Putting in equation (1)

$$L^{-1}\left\{\frac{s^3-3s^2+6s-4}{(s^2-2s+2)^2}\right\} = e^t\cos t + te^t \sin t.$$

Example

Find the convolution $u(t-3) * e^{-2t}$

Ans:-

we have,

$$u(t-3) * e^{-2t} = \int_0^t u(\tau-3)\, e^{-2(t-\tau)}\, d\tau$$

$$= \int_0^t u(\tau-3)\, e^{-2t} e^{2\tau}\, d\tau$$

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$$= e^{-2t} \int_0^t U(\tau-3)\, e^{2\tau} d\tau$$

$$= e^{-2t} \left[ \int_0^3 U(\tau-3)\, e^{2\tau} d\tau + \int_3^t U(\tau-3)\, e^{2\tau} d\tau \right]$$

$$= e^{-2t} \left[ 0 + \int_3^t e^{2\tau} d\tau \right]$$

$$= e^{-2t} \left[ \frac{e^{2\tau}}{2} \right]_3^t$$

$$= \frac{1}{2} e^{-2t} \left( e^{2t} - e^{6} \right)$$

$$= \frac{1}{2} \left( 1 - e^{-2t+6} \right) U(t-3)$$

$$= \frac{1}{2} \left( 1 - e^{-2(t-3)} \right) U(t-3) .$$

Example

Solve the initial value problem

$$y'' + 4y = r(t), \quad r(t) = \begin{cases} 1, & 0 < t < 1 \\ 0, & t > 1 \end{cases} \quad y(0) = 1, \; y'(0) = 0$$

(1)

Ans:- An unit step function $r(t)$ can be written as

$$r(t) = 1\, U(t-0) - 1\, U(t-1)$$

$$L\{r(t)\} = L\{U(t-0)\} - L\{U(t-1)\}$$

$$= \frac{e^{-0s}}{s} - \frac{e^{-1s}}{s}$$

$$= \frac{1}{s} - \frac{e^{-s}}{s} .$$

Let $L\{y\} = Y(s)$

Taking the Laplace transform of both sides of equation (1)

$$L\{y''\} + 4L\{y\} = L\{r(t)\}$$

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$$\Rightarrow s^2 L\{y\} - s\,y(0) - y'(0) + 4L\{y\} = \frac{1}{s} - \frac{e^{-s}}{s}$$

$$\Rightarrow s^2 Y(s) - s \cdot 1 - 0 + 4Y(s) = \frac{1}{s} - \frac{e^{-s}}{s}$$

$$\Rightarrow (s^2+4)\,Y(s) = s + \frac{1}{s} - \frac{e^{-s}}{s}$$ which is known as the subsidary equation.

$$\Rightarrow Y(s) = \frac{s}{s^2+4} + \frac{1}{s(s^2+4)} - \frac{e^{-s}}{s(s^2+4)}$$

Taking the inverse Laplace transform of both sides.

$$L^{-1}\{Y(s)\} = L^{-1}\left\{\frac{s}{s^2+4}\right\} + L^{-1}\left\{\frac{1}{(s^2+4)s}\right\} - L^{-1}\left\{\frac{e^{-s}}{(s^2+4)s}\right\} \quad \text{——(1)}$$

we have,

$$L^{-1}\left\{\frac{s}{s^2+4}\right\} = \cos 2t$$

$$L^{-1}\left\{\frac{1}{(s^2+4)s}\right\}$$

we have, $\frac{1}{(s^2+4)s} = \frac{1}{s^2+4} \cdot \frac{1}{s}$

Let $F(s) = \frac{1}{s^2+4}$ $\quad G(s) = \frac{1}{s}$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s^2+4}\right\} = \frac{1}{2}\sin 2t = f(t)$$

$$L^{-1}\{G(s)\} = L^{-1}\left\{\frac{1}{s}\right\} = 1 = g(t)$$

by convolution theorem

$$L^{-1}\left\{\frac{1}{(s^2+4)s}\right\} = \int_0^t \frac{1}{2}\sin 2\tau \cdot 1\, d\tau$$

$$= \frac{1}{2}\left[-\frac{\cos 2\tau}{2}\right]_0^t$$

$$= -\frac{1}{4}\left[\cos 2t - 1\right]$$

$$= \frac{1}{4}(1 - \cos 2t)$$

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$$L^{-1}\left\{ \frac{e^{-s}}{(s^2+4)\,s} \right\}$$

Let $F(s) = \dfrac{1}{(s^2+4)\,s}$

$$L^{-1}\{F(s)\} = L^{-1}\left\{ \frac{1}{(s^2+4)\,s} \right\} = \frac{1}{4}(1-\cos 2t) = f(t)$$

Thus by S.S.T

$$L^{-1}\left\{ \frac{e^{-s}}{(s^2+4)\,s} \right\} = f(t-1)\,u(t-1)$$

$$= \frac{1}{4}(1-\cos 2(t-1))\,u(t-1)$$

Putting in equation (2)

$$y = \cos 2t + \frac{1}{4}(1-\cos 2t) - \frac{1}{4}(1-\cos 2(t-1))\,u(t-1)$$

is the solution of the given DE.

Example

Find the inverse Laplace transform of

$$\frac{s^2+1}{s^2-3s+2}.$$

Ans:

$$\begin{array}{r|l|l} s^2-3s+2 & s^2+1 & 1 \\ & \underline{s^2-3s+2} & \\ & -\ +\ - & \\ & 3s-1 & \end{array}$$

Thus

$$\frac{s^2+1}{s^2-3s+2} = 1 + \frac{3s-1}{s^2-3s+2} \quad \text{———} (1)$$

We have, by Partial fraction

$$\frac{3s-1}{s^2-3s+2} = \frac{3s-1}{(s-1)(s-2)} = \frac{A}{s-1} + \frac{B}{s-2}$$

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$\Rightarrow \quad 3s-1 = A(s-2) + B(s-1)$

Put $s=1, \quad 2 = -A \Rightarrow A = -2$

Put $s=2, \quad 5 = B \Rightarrow B = 5$

Thus, $\dfrac{3s-1}{s^2-3s+2} = \dfrac{-2}{s-1} + \dfrac{5}{s-2}$

Putting in equation (1)

$$\frac{s^2+1}{s^2-3s+2} = 1 - \frac{2}{s-1} + \frac{5}{s-2}$$

$$L^{-1}\left\{\frac{s^2+1}{s^2-3s+2}\right\} = L^{-1}\{1\} - 2L^{-1}\left\{\frac{1}{s-1}\right\} + 5L^{-1}\left\{\frac{1}{s-2}\right\}$$

$$= \delta(t) - 2e^{t} + 5e^{2t}.$$

## Example

Find the inverse Laplace transform of

$$\frac{s^2-\pi^2}{(s^2+\pi^2)^2}$$

Ans: We have,

$$\frac{s^2-\pi^2}{(s^2+\pi^2)^2} = \frac{s^2+\pi^2-2\pi^2}{(s^2+\pi^2)^2} = \frac{s^2+\pi^2}{(s^2+\pi^2)^2} - \frac{2\pi^2}{(s^2+\pi^2)^2}$$

$$= \frac{1}{s^2+\pi^2} - \frac{2\pi^2}{(s^2+\pi^2)^2}$$

$$L^{-1}\left\{\frac{s^2-\pi^2}{(s^2+\pi^2)^2}\right\} = L^{-1}\left\{\frac{1}{s^2+\pi^2}\right\} - 2L^{-1}\left\{\frac{\pi^2}{(s^2+\pi^2)^2}\right\}$$

$$= \frac{1}{\pi}\sin\pi t - 2\sin\pi t * \sin\pi t$$

$$= \frac{1}{\pi}\sin\pi t - \int_0^t 2\sin\pi\tau \sin\pi(t-\tau)\,d\tau$$

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$$= \frac{1}{\pi}\sin \pi t - \int_0^t [\cos(\pi\tau - \pi t + \pi\tau) - \cos(\pi\tau + \pi t - \pi\tau)]\,d\tau$$

$$= \frac{1}{\pi}\sin \pi t - \int_0^t [\cos(2\pi\tau - \pi t) - \cos \pi t]\,d\tau$$

$$= \frac{1}{\pi}\sin \pi t - \left[\frac{1}{2\pi}\sin(2\pi\tau - \pi t) - \tau \cos \pi t\right]_0^t$$

$$= \frac{1}{\pi}\sin \pi t - \left[\frac{1}{2\pi}\sin \pi t - t\cos \pi t + \frac{1}{2\pi}\sin \pi t - 0\right]$$

$$= \frac{1}{\pi}\sin \pi t - \frac{1}{\pi}\sin \pi t + t\cos \pi t$$

$$= t\cos \pi t .$$

Example

Solve the initial value problem

$$y'' + 4y' + 5y = \delta(t-1) \qquad y(0) = 0,\ y'(0) = 3$$

(1)

Ans: Let $L\{y\} = Y(s)$

Taking the Laplace transform of both sides of (1)

$$L\{y''\} + 4L\{y'\} + 5L\{y\} = L\{\delta(t-1)\}$$

$$\Rightarrow s^2 L\{y\} - sy(0) - y'(0) + 4[sL\{y\} - y(0)] + 5L\{y\} = L\{\delta(t-1)\}$$

$$\Rightarrow s^2 Y(s) - s\cdot 0 - 3 + 4sY(s) - 4\cdot 0 + 5Y(s) = e^{-s}$$

$$\Rightarrow (s^2 + 4s + 5)\,Y(s) = 3 + e^{-s}$$ which is known as the subsidary equation

$$\Rightarrow Y(s) = \frac{3}{s^2+4s+5} + \frac{e^{-s}}{s^2+4s+5}$$

Taking the inverse Laplace transform of both sides.

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$$L^{-1}\{Y(s)\} = 3L^{-1}\left\{\frac{1}{s^2+4s+5}\right\} + L^{-1}\left\{\frac{e^{-s}}{s^2+4s+5}\right\} \quad \text{---(2)}$$

$$L^{-1}\left\{\frac{1}{s^2+4s+5}\right\} = L^{-1}\left\{\frac{1}{(s+2)^2+1}\right\} = e^{-2t}\sin t \quad \text{(by F.S.T)}$$

$$L^{-1}\left\{\frac{e^{-s}}{s^2+4s+5}\right\}$$

Let $F(s) = \frac{1}{s^2+4s+5}$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{(s+2)^2+1}\right\} = e^{-2t}\sin t \quad \text{(by F.S.T)}$$
$$= f(t)$$

By S.S.T

$$L^{-1}\left\{\frac{e^{-s}}{s^2+4s+5}\right\} = f(t-1)\,u(t-1)$$
$$= e^{-2(t-1)}\sin(t-1)\,u(t-1).$$

Example

Solve the initial value problem

$$y'' + 5y' + 6y = u(t-1) + \delta(t-2) \quad \text{---(1)}$$
$$y(0) = 0$$
$$y'(0) = 1$$

Ans: Let $L\{y\} = Y(s)$

Taking the Laplace transform of both sides of (1)

$$L\{y''\} + 5L\{y'\} + 6L\{y\} = L\{u(t-1)\} + L\{\delta(t-2)\}$$

$$\Rightarrow s^2L\{y\} - sy(0) - y'(0) + 5sL\{y\} - 5y(0) + 6L\{y\}$$
$$= L\{u(t-1)\} + L\{\delta(t-2)\}$$

$$\Rightarrow s^2Y(s) - s\cdot 0 - 1 + 5sY(s) - 5\cdot 0 + 6Y(s)$$
$$= \frac{e^{-s}}{s} + e^{-2s}$$

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$\Rightarrow (s^2+5s+6)\,Y(s) = 1 + e^{-2s} + \frac{e^{-s}}{s}$ which is known as the subsidary equation

$\Rightarrow Y(s) = \frac{1}{s^2+5s+6} + \frac{e^{-2s}}{s^2+5s+6} + \frac{e^{-s}}{s(s^2+5s+6)}$

Taking the inverse Laplace transform of both sides

$$L^{-1}\{Y(s)\} = L^{-1}\left\{\frac{1}{(s+2)(s+3)}\right\} + L^{-1}\left\{\frac{e^{-2s}}{(s+2)(s+3)}\right\} + L^{-1}\left\{\frac{e^{-s}}{s(s+2)(s+3)}\right\} \quad \text{---(1)}$$

$L^{-1}\left\{\frac{1}{(s+2)(s+3)}\right\}$

We have,

$$L^{-1}\left\{\frac{1}{(s+2)(s+3)}\right\} = L^{-1}\left\{\frac{1}{s+2} - \frac{1}{s+3}\right\}$$

$$= e^{-2t} - e^{-3t}$$

$L^{-1}\left\{\frac{e^{-2s}}{(s+2)(s+3)}\right\}$

Here, $F(s) = \frac{1}{(s+2)(s+3)}$

$$L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{(s+2)(s+3)}\right\} = e^{-2t} - e^{-3t}$$

By S.S.T

$$L^{-1}\left\{\frac{e^{-2s}}{(s+2)(s+3)}\right\} = f(t-2)\cup(t-2)$$

$$= \left[e^{-2(t-2)} - e^{-3(t-2)}\right]\cup(t-2).$$

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$$L^{-1}\left\{ \frac{e^{-s}}{s(s+2)(s+3)} \right\}$$

Here, $F(s) = \frac{1}{s(s+2)(s+3)}$ —(1)

$$L^{-1}\{F(s)\} = L^{-1}\left\{ \frac{1}{s} \cdot \frac{1}{(s+2)(s+3)} \right\} = \int_0^t \left( e^{-2\tau} - e^{-3\tau} \right) d\tau$$

$$= \left[ \frac{e^{-2\tau}}{-2} - \frac{e^{-3\tau}}{-3} \right]_0^t$$

$$= -\frac{1}{2} e^{-2t} + \frac{1}{3} e^{-3t} + \frac{1}{2} - \frac{1}{3}$$

$$= \frac{1}{3} e^{-3t} - \frac{1}{2} e^{-2t} + \frac{1}{6}$$

By S.S.T

$$L^{-1}\left\{ \frac{e^{-s}}{s(s+2)(s+3)} \right\} = f(t-1)\, U(t-1)$$

$$= \left[ \frac{1}{3} e^{-3(t-1)} - \frac{1}{2} e^{-2(t-1)} + \frac{1}{6} \right] U(t-1)$$

Putting in equation (1)

$$y = e^{-2t} - e^{-3t} + \left[ e^{-2(t-2)} - e^{-3(t-2)} \right] U(t-2)$$

$$+ \left[ \frac{1}{3} e^{-3(t-1)} - \frac{1}{2} e^{-2(t-1)} + \frac{1}{6} \right] U(t-1)$$

is the solution of the given DE.

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